LEONHARDI EULERI OPERA OMNIA

SUB AUSPICIIS SOCIETATIS SCIENTIARUM NATURALIUM HELVETICAE

EDENDA CURAVERUNT

FERDINAND RUDIO · ADOLF KRAZER · PAUL STÄCKEL

SERIES I · OPERA MATHEMATICA · VOLUMEN XIII

LEONHARDI EULERI

INSTITUTIONES CALCULI INTEGRALIS

EDIDERUNT

FRIEDRICH ENGEL ET LUDWIG SCHLESINGER

VOLUMEN TERTIUM

LIPSIAE ET BEROLINI

TYPIS ET IN AEDIBUS B. G. TEUBNERI

MCMXIV

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VORWORT DER HERAUSGEBER

Im zweiten Bande der *Institutiones calculi integralis* bilden die Kapitel über die linearen gewöhnlichen Differentialgleichungen zweiter Ordnung eine ausgedehnte Theorie von großer Tragweite, der gegenüber alles andre, was die beiden ersten Bände über die gewöhnlichen Differentialgleichungen bringen, weit zurücktritt. Der vorliegende dritte Band, der dreizehnte der ersten Serie der *Opera omnia*, enthält keine Theorie, die sich an Umfang und Bedeutung mit der eben erwähnten messen könnte. Aber da EULER hier auf dem weiten Gebiete der partiellen Differentialgleichungen alle die Probleme zusammenstellt, die damals und zwar hauptsächlich doch durch ihn selbst bereits gelöst waren, so bringt auch dieser Band vieles, was heute noch wichtig ist: Keime künftiger Entwickelungen, aus denen inzwischen selbständige Theorien hervorgegangen sind.

Es ist eine merkwürdige, wenig beachtete Tatsache, daß EULER, der für die gewöhnlichen Differentialgleichungen in zwei Veränderlichen so viel getan hat, in seinem *Calculus integralis* über die simultanen Systeme von gewöhnlichen Differentialgleichungen in drei und mehr Veränderlichen fast gar nichts zu sagen hat. Nur an einer später zu erwähnenden Stelle bildet ein System dieser Art in drei Veränderlichen den Gegenstand einer besonderen Untersuchung. In der systematischen Einteilung des *Calculus integralis*, die er an die Spitze seines Werkes stellt[1]), ist immer nur von einer unbekannten Funktion die Rede, und obwohl er durch die von ihm behandelten Probleme oft genug auf simultane Systeme geführt wird, die zu integrieren sind[2]), macht er doch nirgends eine Andeutung, daß auch eine selbständige Theorie derartiger Systeme möglich und nötig sei.

Hiermit stimmt überein, daß EULER auch in der Theorie der partiellen Differentialgleichungen immer nur den Fall einer unbekannten Funktion behandelt, die einer gegebenen Gleichung genügen soll. Die Betrachtung mehrerer Gleichungen, die gleichzeitig zu befrie-

1) Vol. I, § 13; *LEONHARDI EULERI Opera omnia*, series I, vol. 11, p. 8.

2) A. a. O. § 493—529, im darauf folgenden Bande § 1124 und im vorliegenden § 349, 367, 373—377.

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SERIES PRIMA
OPERA MATHEMATICA
VOLUMEN TERTIUM DECIMUM

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digen wären, schließt er aus. Er erklärt in § 444, das Problem sei durch eine Gleichung vollständig bestimmt; durch zwei Gleichungen sei es überbestimmt und lasse eine Lösung nur in gewissen Fällen zu, in denen die eine Gleichung sozusagen schon in der andern enthalten sei.

Den ganzen im vorliegenden Bande zu behandelnden Stoff zerlegt EULER in zwei Teile. In dem ersten, bei weitem ausgedehnteren, behandelt er in drei Abschnitten die Bestimmung einer Funktion von zwei unabhängigen Veränderlichen, während er den zweiten, über die Funktionen dreier Veränderlicher, nicht einmal in Abschnitte gliedert (§ 429), denn er könne, wie er selbst erklärt (§ 468), hier doch kaum über die ersten Anfänge hinausgehen.

Als Einleitung zu den partiellen Differentialgleichungen erster Ordnung mit zwei unabhängigen Veränderlichen (Teil I, Abschn. I) dient ein Kapitel über die Gleichungen von der Form $Pdx + Qdy + Rdz = 0$, wo P, Q, R Funktionen von x, y, z sind. EULER beschränkt sich hier von vornherein auf die Betrachtung des Falles, wo diese Gleichung aus einer endlichen Gleichung von der Form $f(x, y, z) =$ Const. durch Differentiation und nachherige Division mit einer Funktion von x, y, z entstanden ist und also durch Multiplikation mit einem Faktor integrabel wird. Daß die hierfür notwendige Bedingung, das identische Verschwinden des Ausdrucks $LP + MQ + NR$ (§ 6), auch hinreichend ist, betrachtet er infolgedessen als selbstverständlich, wenigstens findet sich nirgends eine Andeutung von der Notwendigkeit eines Beweises[1]). Bei diesem Ausgangspunkte ist es auch vollständig zu verstehen, daß EULER Gleichungen, für die $LP + MQ + NR$ nicht identisch verschwindet, ganz von der Betrachtung ausschließt und sie immer von neuem als „absurdas" und „nihil plane significantes" bezeichnet (§ 2, 6, 7, 11). Um so nötiger ist es, darauf hinzuweisen, daß EULER sehr wohl wußte, die Gleichung $LP + MQ + NR = 0$ könne unter Umständen eine Lösung der Differentialgleichung liefern[2]).

In seinen Briefen an W. J. G. KARSTEN (1732—1787) spricht sich EULER in ähnlicher Weise wie hier über die Gleichungen von der Form $Pdx + Qdy + Rdz = 0$ aus[3]), er fügt aber hinzu, daß auch eine nicht integrable Gleichung dieser Art, wenn noch eine zweite Differentialgleichung erster Ordnung zwischen x, y, z hinzukommt, immer auf ein lösbares Problem führt. Auf den Gedanken ist er jedoch nicht gekommen, daß man eben

1) Den ersten Beweis hat LAGRANGE gegeben: *Sur l'intégration des équations à différences partielles du premier ordre.* Nouv. mém. de l'acad. d. sc. de Berlin (1772), 1774, p. 353; *Oeuvres de LAGRANGE*, t. III, p. 549.

2) Vgl. *Institutiones calculi differentialis*, partis prioris § 324 et 325; *LEONHARDI EULERI Opera omnia*, series I, vol. 10, p. 211.

3) Es sind die Briefe vom 6. Juli und 5. August 1760, abgedruckt in der Allgemeinen Monatsschrift für Wissenschaft und Literatur, Braunschweig 1854, p. 328—333; *LEONHARDI EULERI Opera omnia*, series III.

deshalb auch der Gleichung $Pdx + Qdy + Rdz = 0$ an sich einen Sinn beilegen könne. Das zu tun war MONGE vorbehalten, dessen Auffassung dann durch J. F. PFAFF und später durch SOPHUS LIE eine so außerordentliche Bedeutung für die partiellen Differentialgleichungen erster Ordnung gewonnen hat.

Am Schlusse des einleitenden Kapitels (§ 31) spricht es EULER mit voller Klarheit aus, daß eine partielle Differentialgleichung erster Ordnung zwischen x, y, z als gelöst anzusehen ist, wenn es gelingt, sie auf die Integration gewöhnlicher Differentialgleichungen in zwei Veränderlichen zurückzuführen. Dabei wird, wie er später (§ 113) noch gelegentlich erwähnt, die Auflösung von Gleichungen beliebigen Grades, ja sogar die von transzendenten, ein für allemal als zugestanden angenommen. Weil ihm nun aber die Zurückführung auf gewöhnliche Differentialgleichungen nicht allgemein gelungen ist, so teilt er in § 32 die zu behandelnden partiellen Differentialgleichungen rein äußerlich nach ihrer Schwierigkeit ein, also danach, ob von den Größen $\left(\frac{dz}{dx}\right) = p$ und $\left(\frac{dz}{dy}\right) = q$ nur eine auftritt oder beide, und im zweiten Falle danach, ob von den Größen x, y, z nur eine, zwei oder alle drei vorkommen.

Hierdurch ist der Inhalt der Kapitel II—VI des I. Abschnitts bedingt. Näher darauf einzugehen ist unnötig; nur ein paar allgemeine Bemerkungen dürften am Platze sein.

Es stellt sich sofort heraus (§ 33), daß die allgemeine Lösung einer Differentialgleichung der hier betrachteten Art eine willkürliche Funktion einer veränderlichen Größe enthält, und EULER wird nicht müde, hervorzuheben, daß man unter diesen willkürlichen Funktionen keineswegs bloß „functiones continuas" zu verstehen habe, also „functiones per operationes analyticas conflatas", daß man sie sich vielmehr ebensogut als „functiones discontinuas" denken könne, das heißt als solche, wie sie eine „curva quaecunque libero manus ductu descripta" liefert, und sei auch die Kurve „maxime irregularis et ex pluribus partibus diversarum curvarum conflata" (vgl. besonders § 37, 299 und 301)[1]. Er gedenkt dabei auch (§ 37) ohne Namensnennung seines bekannten Streites mit D'ALEMBERT über die allgemeine Lösung der partiellen Differentialgleichung für die schwingenden Saiten, eines Streites, der bekanntlich später von DANIEL BERNOULLI neu aufgenommen wurde und das ganze achtzehnte Jahrhundert hindurch die Mathematiker in Aufregung versetzt hat.

Zu erwähnen ist sodann, daß EULER immer mit der Gleichung $dz - pdx - qdy = 0$ operiert, die er auf alle möglichen Arten integrabel zu machen sucht. Es ist in hohem Grade bemerkenswert, wie sehr bei ihm schon die fünf Veränderlichen x, y, z, p, q als gleich-

1) Vgl. auch die Abhandlung 322 des ENESTRÖMSCHEN Verzeichnisses: *De usu functionum discontinuarum in analysi*, Novi comment. acad. sc. Petrop. **11** (1765), 1767, p. 3; *LEONHARDI EULERI Opera omnia*, series I, vol. 23.

berechtigt erscheinen. Für ihn ist es daher nicht im geringsten störend, daß sich in allen schwierigeren Fällen die allgemeine Lösung nur durch zwei oder gar drei Gleichungen zwischen x, y, z, p, q darstellen läßt (§ 84, 85 und 87), denn man kann ja zwei beliebige unter diesen fünf Größen als unabhängige Veränderliche benutzen. Unter diesem Gesichtspunkte ist es für ihn etwas ganz selbstverständliches, die Gleichung $dz = \int(p\,dx + q\,dy)$ in der Form

$$z = px + qy - \int(x\,dp + y\,dq) \quad \text{oder} \quad z = px + \int(q\,dy - x\,dp)$$

zu schreiben (§ 82, 109) und also an Stelle von z die Größen $z - px - qy$ oder $z - px$ als neue unbekannte Funktionen einzuführen. Was er hier macht, ist nichts andres als die Ausführung gewisser Berührungstransformationen im Sinne von LIE; nur die formelle Vereinfachung fehlt noch, daß die benutzten Operationen ausdrücklich als Transformationen in den Veränderlichen x, y, z, p, q geschrieben werden. Es ist daher nicht richtig, daß man, wie es meistens geschieht, die dem ersten Falle entsprechende Transformation

$$z_1 = z - xp - yq, \quad x_1 = p, \quad y_1 = q, \quad p_1 = -x, \quad q_1 = -y,$$

die ja als analytischer Ausdruck der Dualität von ganz besonderer Wichtigkeit ist, nach LEGENDRE benennt und die dem zweiten Falle entsprechende

$$z_1 = z - px, \quad x_1 = p, \quad p_1 = -x, \quad y_1 = y, \quad q_1 = q$$

nach AMPÈRE. Man sollte vielmehr, wie es LIE in seinen ersten Arbeiten getan hat[1]), beide Transformationen und ihre Verallgemeinerungen auf beliebig viele Veränderliche schlechthin als EULERSCHE Transformationen bezeichnen, um so mehr, als EULER selbst solche Transformationen auch auf partielle Differentialgleichungen erster Ordnung in vier Veränderlichen angewendet hat (§ 485, 489).

Die Darstellung der allgemeinen Lösung einer partiellen Differentialgleichung erster Ordnung durch mehrere Gleichungen bringt es mit sich, daß EULER häufig, wenn er die Integration auf verschiedene Arten ausführt, ganz verschieden aussehende Formen für die allgemeine Lösung erhält; doch haben diese verschiedenen Formen immer das gemein, daß in jeder eine willkürliche Funktion einer veränderlichen Größe und außerdem die Ableitung dieser willkürlichen Funktion auftritt. EULER klärt diese Erscheinung auf, indem er jedesmal zeigt, wie die eine Form auf die andre zurückgeführt werden kann (§ 118, 119, 121, 122, 177). Die Umformungen, die er da anwendet, sind im Grunde wieder Berührungstransformationen, diesmal aber Berührungstransformationen der Ebene, und zwar treten deren

1) Vgl. die Abhandlung von LIE: *Allgemeine Theorie partieller Differentialgleichungen 1. O.*, Forhandlinger i Videnskabsselskabet i Christiania 1874, p. 209 u. 226; *S. LIE, Gesammelte Abhandlungen*, Bd. III, p. 159 u. 174.

mehrere auf von ganz verschiedener Form. Einmal benutzt er auch eine solche Umformung, um die gefundene allgemeine Lösung einer Differentialgleichung von Logarithmen zu befreien und dadurch zu vereinfachen (§ 169); die betreffenden Operationen kommen darauf hinaus, daß an Stelle einer willkürlichen Funktion u von t eine willkürliche Funktion u_1 von t_1 durch die Berührungstransformation

$$t_1 = t\frac{du}{dt}, \quad u_1 = \left(\log t + \frac{a-b}{4ab}\right)t\frac{du}{dt} - u, \quad \frac{du_1}{dt_1} = \log t + \frac{a-b}{4ab}$$

eingeführt wird.

In diesen Entwickelungen EULERS hat man Vorläufer gewisser Untersuchungen JACOBIS zu erblicken. JACOBI hat zuerst allgemein gezeigt, wie man aus einer vollständigen Lösung einer partiellen Differentialgleichung erster Ordnung durch gewisse Umformungen alle andern vollständigen Lösungen ableiten kann[1]), und diese Untersuchungen JACOBIS hinwiederum sind eine der Quellen, aus denen unter den Händen von LIE die allgemeine Theorie der Berührungstransformationen hervorgegangen ist.

Es war EULER nicht gelungen, die allgemeine partielle Differentialgleichung erster Ordnung zwischen x, y, z auf gewöhnliche Differentialgleichungen zurückzuführen, er war sogar nicht einmal imstande, das für die allgemeine lineare Gleichung zu leisten. Der allgemeinste Fall dieser letzteren Gleichungen, den er erledigt (§ 209, 407), hat die Form

$$Z = P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right),$$

wo P und Q beliebige Funktionen von x, y sind und Z eine Funktion von z allein. Doch ist auch diese Erledigung insofern unbefriedigend, als er sie darauf gründet, daß man bei gegebenen P und Q immer M und N so wählen könne, daß der Ausdruck

$$\frac{Mdx + Ndy}{MP + NQ}$$

integrabel werde, während er nicht zeigt, wie diese letztere Aufgabe wirklich gelöst werden kann. Um so auffallender ist es, daß er in der zweiten Abteilung (§ 484) die allgemeine lineare homogene partielle Differentialgleichung erster Ordnung in vier Veränderlichen tatsächlich auf gewöhnliche Differentialgleichungen zurückführt. Er zeigt nämlich, daß die Gleichung

$$L\left(\frac{dv}{dx}\right) + M\left(\frac{dv}{dy}\right) + N\left(\frac{dv}{dz}\right) = 0,$$

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1) Vgl. die nachgelassene Abhandlung von JACOBI: *Ueber die vollständigen Lösungen einer partiellen Differentialgleichung erster Ordnung* in den *Vorlesungen über Dynamik,* herausg. von A. CLEBSCH, Berlin 1866, p. 471; *C. G. J. JACOBIS Gesammelte Werke,* Bd. V, p. 397.

in der L, M, N Funktionen von x, y, z sind, gelöst ist, sobald es gelingt, zwei Systeme von Multiplikatoren E, F und G, H so zu bestimmen, daß die Ausdrücke

$$E\left(dx-\frac{L\,dz}{N}\right)+F\left(dy-\frac{M\,dz}{N}\right),\quad G\left(dx-\frac{L\,dz}{N}\right)+H\left(dy-\frac{M\,dz}{N}\right)$$

integrabel werden, bemerkt aber freilich dazu, das könne wohl anscheinend immer geleistet werden, sei aber eine Aufgabe, die meistens schwieriger ausfalle als die ursprüngliche. Hieraus scheint hervorzugehen, daß EULER den Zusammenhang dieser Aufgabe mit der Integration der gewöhnlichen Differentialgleichungen

$$dx=\frac{L}{N}\,dz,\quad dy=\frac{M}{N}\,dz$$

nicht erkannt hat; sonst würde es ihm wohl auch nicht entgangen sein, daß die allgemeine lineare partielle Differentialgleichung in x, y, z

$$L\left(\frac{dz}{dx}\right)+M\left(\frac{dz}{dy}\right)=N$$

auf dasselbe Integrationsproblem führt. Das Ganze ist ein neuer Beweis dafür, daß die allgemeine Theorie der simultanen gewöhnlichen Differentialgleichungen erster Ordnung EULER noch fern lag.

Daß EULERS Leistungen in der Theorie der partiellen Differentialgleichungen erster Ordnung zwischen x, y, z für die Geschichte dieser Theorie wenig Bedeutung haben, liegt daran, daß seine besondern Methoden fast alle entbehrlich wurden, als es gelang, das allgemeine Problem auf gewöhnliche Differentialgleichungen zurückzuführen, was bekanntlich zuerst LAGRANGE geleistet hat. Bei den partiellen Differentialgleichungen zweiter und höherer Ordnung ist diese Zurückführung überhaupt nicht allgemein möglich; EULER aber war der erste, der große Klassen von Differentialgleichungen angegeben hat, wo sie möglich ist, und deshalb werden seine Untersuchungen hierüber (Abschn. II und III von Teil I, im ganzen 8 Kapitel) in der Geschichte der Theorie dieser Gleichungen immer an erster Stelle zu nennen sein. Sie müssen daher auch hier etwas näher besprochen werden.

Wir erwähnen zuerst eine Bemerkung, die EULER bei der Behandlung der Differentialgleichungen der schwingenden Saiten gelegentlich (§ 301) macht und auf die er auch später (§ 420) zurückkommt. Er hebt nämlich hervor, daß, im Gegensatze zur Differentialgleichung der schwingenden Saite, die allgemeine Lösung der Gleichung

$$\left(\frac{ddz}{dy^2}\right)+aa\left(\frac{ddz}{dx^2}\right)=0$$

auf imaginäre Ausdrücke führt, die nur dann die Form $P\pm Q\sqrt{-1}$ erhalten können, wenn die betreffenden willkürlichen Funktionen „per operationes analyticas sunt conflatae, hoc est

continuae", während diese Ausdrücke alle Bedeutung verlieren, wenn die willkürlichen Funktionen nach seiner Bezeichnungsweise „discontinuae" sind. Es war ihm also vollständig klar, daß zwischen den beiden Typen von partiellen Differentialgleichungen zweiter Ordnung, die man heutzutage hyperbolisch und elliptisch nennt, ein Wesensunterschied besteht[1]).

Nun zu den wichtigsten Einzeluntersuchungen.

Da die Differentialgleichung

$$\left(\frac{ddz}{dx\,dy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + Rz + S = 0,$$

in der P, Q, R, S Funktionen von x, y sind, unmittelbar auf eine gewöhnliche Differentialgleichung hinauskommt, wenn R und einer der Koeffizienten P, Q verschwindet, so untersucht EULER, wann die allgemeine Gleichung durch eine Substitution von der Form: $z = ve^V$ in eine integrable Gleichung für v überführbar ist (§ 287—295). Sodann betrachtet er (§ 302—321) andre Formen von linearen partiellen Differentialgleichungen zweiter Ordnung, die dadurch, daß man an Stelle von x und y neue unabhängige Veränderliche t und u einführt, auf die eben gefundenen integrabeln Formen gebracht werden können.

Eine wichtige Rolle in der Geschichte der Theorie spielt die partielle Differentialgleichung zweiter Ordnung

$$(x+y)^2\left(\frac{ddz}{dx\,dy}\right) + m(x+y)\left(\frac{dz}{dx}\right) + m(x+y)\left(\frac{dz}{dy}\right) + nz = 0,$$

die von § 322 an behandelt wird; in etwas allgemeinerer Form, ohne daß die beiden mittleren Koeffizienten als gleich angenommen werden, ist sie später von LAPLACE behandelt worden[2]) und DARBOUX äußert über sie: „De toutes les équations que nous aurions pu choisir, il n'en est aucune dont l'étude présente plus d'intérêt et puisse fournir autant d'indications précieuses sur l'intégration des équations linéaires les plus générales."[3])

Indem er die Gleichung durch eine unendliche Reihe befriedigt, die nach Potenzen von $x+y$ fortschreitet, gelangt EULER zu den Fällen, in denen die allgemeine Lösung in endlicher geschlossener Form darstellbar ist, und erhält so Differentialgleichungen, in deren allgemeiner Lösung eine willkürliche Funktion einer veränderlichen Größe und die Ableitungen dieser Funktion bis zu einer beliebigen Ordnung auftreten. EULER hebt das selbst als etwas Neues und Merkwürdiges hervor (§ 327, am Schlusse). Sodann untersucht er (§ 330), wann die allgemeinere Gleichung

$$\left(\frac{ddz}{dy^2}\right) - QQ\left(\frac{ddz}{dx^2}\right) + R\left(\frac{dz}{dy}\right) + S\left(\frac{dz}{dx}\right) + Tz = 0$$

1) Vgl. etwa Encyklopädie der Math. Wiss. Art. II A 7c (A. SOMMERFELD) Nr. 8.

2) Vgl. die Note p. 212.

3) G. DARBOUX, *Leçons sur la théorie générale des surfaces*, II. partie, Paris 1889, p. 55.

durch Einführung neuer Veränderlicher auf die hier betrachtete Form gebracht werden kann, und gelangt so zu unendlich vielen integrablen Gleichungen, die unter anderm für die Schwingungen der Saiten von veränderlicher Dicke und für die Fortpflanzung des Schalles von Bedeutung sind (§ 333—345). Die auffallende Analogie der gefundenen integrablen Fälle mit den aus der Theorie der RICCATISCHEN Gleichung bekannten gibt ihm Veranlassung, den innern Grund für diese Analogie zu entwickeln (§ 346). Durch die Substitution $z = e^{\alpha y} v$, wo v eine Funktion von x allein ist, erhält man nämlich für v eine gewöhnliche Differentialgleichung zweiter Ordnung, die unmittelbar auf die RICCATISCHE zurückführbar ist. Man sieht hieraus, daß auch das Verfahren, eine Lösung von der Form $z = \varphi(x)\chi(y)$ zu suchen, auf EULER zurückgeht; in dieser allgemeineren Fassung tritt es übrigens an einer späteren Stelle auch noch auf (§ 414).

Sehr merkwürdig ist das Verfahren, das EULER in § 349 entwickelt, um aus einer Gleichung von der Form

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv,$$

wo F, G, H Funktionen von x sind, neue Gleichungen abzuleiten, die integriert werden können, wenn jene integrabel ist. Er setzt nämlich

$$z = M\left(\frac{dv}{dx}\right) + Nv,$$

wo M und N ebenfalls nur x enthalten, und bestimmt M und N so, daß für z eine Differentialgleichung von ähnlicher Form herauskommt, wie die für v angenommene. Wir haben hier die ersten Anfänge der berühmten von LAPLACE geschaffenen Transformationstheorie der linearen homogenen partiellen Differentialgleichungen zweiter Ordnung.

Setzt man $N = Ms$, so bleibt M beliebig, während s einer Differentialgleichung von der Form

$$\frac{ds}{dx} = \alpha(x) + \beta(x)s + \gamma(x)s^2,$$

also nach unserer jetzigen Ausdrucksweise einer allgemeinen RICCATISCHEN Gleichung genügen muß. Die Behandlung einiger besonderer Fälle veranlaßt EULER zur Einschaltung von Entwickelungen über lineare homogene gewöhnliche Differentialgleichungen zweiter Ordnung, die die betreffenden Untersuchungen des zweiten Bandes der *Institutiones* vervollständigen (§ 353, 359—366).

Erwähnenswert ist es, daß hierbei in § 353 die Differentialgleichung auftritt, die später nach BESSEL benannt worden ist. Ferner findet man in § 364 und 365 Differentialgleichungen, die durch die Substitution $x = \sin\omega$ in diejenigen übergehen, denen die Koeffizienten der Entwickelung des Ausdrucks $(1 - 2\alpha x + \alpha^2)^{-\nu}$ nach Potenzen von α Genüge leisten. In diesen Koeffizienten sind bekanntlich als besondere Fälle auch die LEGENDRESCHEN

Polynome enthalten, die in der Lehre von den Kugelfunktionen eine Rolle spielen[1]). Entwickelungen des erwähnten Ausdrucks für $x = \cos\varphi$ nach den Cosinus der Vielfachen von φ treten zuerst in Abhandlungen EULERS zur Störungstheorie auf und später noch wiederholt in andern Arbeiten, unabhängig von der Störungstheorie[2]). Sie haben dann den Gegenstand ausgedehnter Untersuchungen der hervorragendsten Mathematiker des 18. Jahrhunderts gebildet; knüpft doch auch GAUSS in seiner Abhandlung über die hypergeometrische Reihe (1812) unmittelbar an diese Entwickelungen an[3]).

In § 367 verallgemeinert EULER die vorhin besprochene Untersuchung noch, indem er die Substitution

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv$$

anwendet; es stellt sich jedoch heraus, daß genau dasselbe durch zwei nacheinander ausgeführte Substitutionen von der früheren Form, wo M beide Male gleich 1 gesetzt wird, erreicht werden kann. Das ganze Problem hängt ab von der Integration eines simultanen Systems von Gleichungen erster Ordnung in drei Veränderlichen, aber dieses System ist wegen seiner besondern Beschaffenheit auf zwei gewöhnliche Differentialgleichungen erster Ordnung in je zwei Veränderlichen zurückführbar. Es ist das die einzige Stelle des *Calculus integralis*, wo EULER auf solche simultane Systeme wirklich eingeht (§ 367—378).

Erwähnung verdienen endlich noch die in § 403—407 behandelten Probleme, in denen EULER partielle Differentialgleichungen zweiter und dritter Ordnung betrachtet, die durch die Substitution $z = e^{\alpha x}v$ eine integrable Form annehmen.

Der zweite Teil (§ 430—510), der sich mit der Bestimmung der Funktionen dreier Veränderlicher beschäftigt, enthält im Vergleich mit dem ersten wenig Bemerkenswertes. Ich möchte hier nur auf die elegante Art hinweisen, in der EULER eine beliebige Gleichung zwischen $\left(\frac{dv}{dx}\right)$, $\left(\frac{dv}{dy}\right)$, $\left(\frac{dv}{dz}\right)$ integriert (§ 489), und außerdem auf die Untersuchungen über die Integration linearer homogener Differentialgleichungen beliebiger Ordnung mit konstanten Koeffizienten (§ 492—509).

In einem Briefe, den EULER am 9. (20.) Januar 1767 von Petersburg aus an LAGRANGE gerichtet hat, heißt es[4]):

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1) Vgl. z. B. Encyklopädie der Math. Wiss. Art. II A 10 (A. WANGERIN), Nr. 31, p. 730.

2) Vgl. H. BURKHARDT, *Entwicklungen nach oszillirenden Funktionen,* Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. X, Heft II, § 17 und 19 (Leipzig 1908); ferner *Institutionum calculi integralis* vol. I, § 279, 292, *LEONHARDI EULERI Opera omnia*, series I, vol. 11, p. 165, 176.

3) C. F. GAUSS, *Werke*, Bd. III, p. 128.

4) L. EULERI *Opera postuma* 1, Petropoli 1862, p. 569; *Oeuvres de LAGRANGE*, t. XIV, p. 210; *LEONHARDI EULERI Opera omnia,* series III.

„Dès mon arrivée ici, l'Académie Impériale a bien voulu se charger de l'impression de mon Ouvrage sur le Calcul intégral, qui a déjà avancé assez bien; mais, comme il y en aura trois Volumes *in quarto*, il faudra attendre encore plus qu'un an, avant que tout soit achevé. Le troisième Volume renferme la nouvelle partie du Calcul intégral dont le public sera toujours redevable à votre sagacité, et j'espère que, par vos soins, cette partie que je n'ai fait qu'ébaucher sera bientôt portée à un plus haut point de perfection."

„La nouvelle partie du Calcul intégral" ist offenbar nichts anderes als der *Calculus variationum*, den EULER in dem ersten Anhange des vorliegenden Bandes dargestellt hat (p. 369–469).[1] Beigegeben ist dann noch ein *Supplementum*, das die schon im ersten Teile der *Institutiones* (§ 580–649) untersuchten Differentialgleichungen, die auf die Additionstheoreme der zyklometrischen und der elliptischen Integrale führen, von einem neuen Gesichtspunkte aus behandelt. Die im ersten Teile gegebene Darstellung war eine Wiedergabe des Verfahrens, zu dem EULER gelangt war, als er durch die Entdeckung FAGNANOS veranlaßt worden war, sich von neuem mit diesen Integralen zu beschäftigen.[2] EULER empfand aber sehr wohl, daß diese Methode der Integration nicht ganz befriedigend ist, weil die Form der algebraischen Integralgleichung nicht abgeleitet, sondern von vornherein angesetzt wird, um aus der Integralgleichung die Differentialgleichung zu gewinnen, der sie genügt. In der Theorie seines integrierenden Faktors glaubte nun EULER ein Mittel zu besitzen, diese indirekte Integrationsmethode durch eine direkte und für alle Fälle gleichmäßig brauchbare zu ersetzen. Es ist ihm in der Tat gelungen, in dem *Supplementum* einen solchen unmittelbaren und einheitlichen Weg zur Integration anzugeben. Aus naheliegenden Annahmen über die Form, die der integrierende Faktor möglicherweise haben wird, leitet er diesen Faktor wirklich ab, und es ist immerhin denkbar, daß jemand, der das algebraische Integral noch nicht kennt, auf diesem Wege zur Integration gelangt. Von der berühmten LAGRANGESCHEN Methode zur Integration der betreffenden Differentialgleichungen kann man das jedenfalls nicht sagen.

* * *

Der im Vorworte zum ersten Bande der *Institutiones* mitgeteilte Plan für die Verteilung des Werkes auf die beiden Herausgeber ist vollständig eingehalten worden, jedoch hat jeder von uns beiden auch die dem andern zugewiesenen Teile bis ins einzelnste genau durchgearbeitet. Daher ist die ganze Ausgabe in allen ihren Teilen als das Ergebnis unsrer gemeinsamen Arbeit zu betrachten, und es versteht sich von selbst, daß Kollege SCHLESINGER auch zu diesem Vorworte Beiträge geliefert hat.

1) Vgl. die Anm. p. 375.

2) Vgl. das Vorwort zu Bd. XX der ersten Serie der *Opera omnia*.

Nachdem nunmehr das Ganze glücklich zu Ende geführt ist, können wir nicht umhin, nochmals der peinlichen Sorgfalt zu gedenken, mit der uns die Herren Redaktoren der Eulerausgabe beim Lesen der Korrekturen unterstützt haben.

Die bei der Herausgabe befolgten Grundsätze sind unverändert geblieben. Erwähnen möchten wir nur, daß wir uns redlich bemüht haben, überall da, wo es nötig schien, Zitate hinzuzufügen, selbstverständlich in erster Linie solche aus EULERS sonstigen Schriften. Diese nicht immer leichte Aufgabe würde noch wesentlich zeitraubender gewesen sein, hätten wir nicht das musterhafte ENESTRÖMSCHE Verzeichnis zur Verfügung gehabt.

Gießen, den 16. Juli 1914.

F. ENGEL

INSTITUTIONUM CALCULI INTEGRALIS

VOLUMEN TERTIUM

INSTITVTIONVM CALCVLI INTEGRALIS

VOLVMEN TERTIVM,

IN QVO METHODVS INVENIENDI FVNCTIONES DVARVM ET PLVRIVM VARIABILIVM, EX DATA RELATIONE DIFFERENTIALIVM CVIVSVIS GRADVS PERTRACTATVR.

VNA CVM APPENDICE DE CALCVLO VARIATIONVM ET SVPPLEMENTO, EVOLVTIONEM CASVVM PRORSVS SINGVLARIVM CIRCA INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM CONTINENTE.

AVCTORE

LEONHARDO EVLERO

ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO ACAD. PETROP. PARISIN. ET LONDIN.

PETROPOLI,

Impensis Academiae Imperialis Scientiarum

1770.

INDEX CAPITUM

IN VOLUMINE TERTIO CONTENTORUM

LIBER POSTERIOR

PARS PRIMA

INVESTIGATIO FUNCTIONUM DUARUM VARIABILIUM EX DATA DIFFERENTIALIUM CUIUSVIS GRADUS RELATIONE

SECTIO PRIMA

DE INVESTIGATIONE DUARUM VARIABILIUM FUNCTIONUM EX DATA DIFFERENTIALIUM PRIMI GRADUS RELATIONE

SECTIO SECUNDA

DE INVESTIGATIONE DUARUM VARIABILIUM FUNCTIONUM EX DATA DIFFERENTIALIUM SECUNDI GRADUS RELATIONE

SECTIO TERTIA

DE INVESTIGATIONE DUARUM VARIABILIUM FUNCTIONUM EX DATA DIFFERENTIALIUM TERTII ALTIORUMQUE GRADUUM RELATIONE

PARS ALTERA

INVESTIGATIO FUNCTIONUM TRIUM VARIABILIUM EX DATA DIFFERENTIALIUM RELATIONE

APPENDIX

DE CALCULO VARIATIONUM

SUPPLEMENTUM

CALCVLI INTEGRALIS
LIBER POSTERIOR.

PARS PRIMA

SEV

INVESTIGATIO FVNCTIONVM DVARVM VARIABILIVM EX DATA DIFFERENTIALIVM CVIVSVIS GRADVS RELATIONE.

SECTIO PRIMA

INVESTIGATIO DVARVM VARIABILIVM FVNCTIONVM EX DATA DIFFERENTIALIVM PRIMI GRADVS RELATIONE.

CAPUT I

DE NATURA AEQUATIONUM DIFFERENTIALIUM QUIBUS FUNCTIONES DUARUM VARIABILIUM DETERMINANTUR IN GENERE

PROBLEMA 1

1. *Si z sit functio quaecunque duarum variabilium x et y, definire indolem aequationis differentialis, qua relatio differentialium dx, dy et dz exprimitur.*

SOLUTIO

Sit

$$Pdx + Qdy + Rdz = 0$$

aequatio relationem differentialium dx, dy et dz exprimens, in qua P, Q et R sint functiones quaecunque ipsarum x, y et z. Ac primo quidem necesse est, ut haec aequatio nata sit ex differentiatione aequationis cuiuspiam finitae, postquam differentiale per quampiam quantitatem fuerit divisum. Dabitur ergo quidam multiplicator, puta M, per quem formula $Pdx + Qdy + Rdz$ multiplicata fiat integrabilis; nisi enim talis multiplicator existeret, aequatio differentialis proposita foret absurda nihilque omnino declararet. Totum ergo negotium huc redit, ut character assignetur, cuius ope huiusmodi aequationes differentiales absurdae nihilque significantes a realibus dignosci queant.

Hunc in finem contemplemur aequationem propositam $Pdx + Qdy + Rdz = 0$ tanquam realem. Sit M multiplicator eam reddens integrabilem, ita ut haec formula

$$MPdx + MQdy + MRdz$$

sit verum differentiale cuiuspiam functionis trium variabilium x, y et z; quae functio si ponatur $= V$, haec aequatio $V = \text{Const.}$ futura sit integrale completum aequationis propositae. Sive igitur x sive y sive z accipiatur constans, singulas has formulas

$$MQdy + MRdz, \quad MRdz + MPdx, \quad MPdx + MQdy$$

seorsim integrabiles esse oportet; unde ex natura differentialium erit

$$\left(\frac{d.\,MQ}{dz}\right) - \left(\frac{d.\,MR}{dy}\right) = 0, \quad \left(\frac{d.\,MR}{dx}\right) - \left(\frac{d.\,MP}{dz}\right) = 0, \quad \left(\frac{d.\,MP}{dy}\right) - \left(\frac{d.\,MQ}{dx}\right) = 0,$$

unde per evolutionem hae tres oriuntur aequationes

$$\text{I.} \quad M\left(\frac{dQ}{dz}\right) + Q\left(\frac{dM}{dz}\right) - M\left(\frac{dR}{dy}\right) - R\left(\frac{dM}{dy}\right) = 0,$$

$$\text{II.} \quad M\left(\frac{dR}{dx}\right) + R\left(\frac{dM}{dx}\right) - M\left(\frac{dP}{dz}\right) - P\left(\frac{dM}{dz}\right) = 0,$$

$$\text{III.} \quad M\left(\frac{dP}{dy}\right) + P\left(\frac{dM}{dy}\right) - M\left(\frac{dQ}{dx}\right) - Q\left(\frac{dM}{dx}\right) = 0;$$

quarum si prima per P, secunda per Q et tertia per R multiplicetur, in summa omnia differentialia ipsius M se tollent et reliqua aequatio per M divisa erit

$$P\left(\frac{dQ}{dz}\right) - P\left(\frac{dR}{dy}\right) + Q\left(\frac{dR}{dx}\right) - Q\left(\frac{dP}{dz}\right) + R\left(\frac{dP}{dy}\right) - R\left(\frac{dQ}{dx}\right) = 0,$$

quae continet characterem aequationes differentiales reales ab absurdis discernentem, et quoties inter quantitates P, Q et R haec conditio locum habet, toties aequatio differentialis proposita

$$Pdx + Qdy + Rdz = 0$$

est realis. Caeterum hic meminisse oportet huiusmodi formulam uncinulis inclusam $\left(\frac{dQ}{dz}\right)$ significare valorem $\frac{dQ}{dz}$, si in differentiatione ipsius Q sola quantitas z ut variabilis tractetur; quod idem de caeteris est tenendum, quae ergo semper ad functiones finitas reducuntur.

COROLLARIUM 1

2. Proposita ergo aequatione differentiali inter tres variabiles

$$Pdx + Qdy + Rdz = 0$$

ante omnia dispiciendum est, utrum character inventus locum habeat necne. Priori casu aequatio erit realis, posteriori vero absurda et nihil plane significans neque unquam ad talem aequationem ullius problematis solutio perducere valet.

COROLLARIUM 2

3. Character inventus etiam hoc modo exprimi potest

$$\left(\frac{PdQ - QdP}{dz}\right) + \left(\frac{QdR - RdQ}{dx}\right) + \left(\frac{RdP - PdR}{dy}\right) = 0,$$

quandoquidem uncinulae non quantitates finitas afficiunt, sed solam differentiationem ad certam variabilem restringunt.

COROLLARIUM 3

4. Simili modo si aequatio haec characterem continens per PQR dividatur, ea hanc formam induet

$$\left(\frac{d.\,l\frac{Q}{P}}{Rdz}\right) + \left(\frac{d.\,l\frac{R}{Q}}{Pdx}\right) + \left(\frac{d.\,l\frac{P}{R}}{Qdy}\right) = 0,$$

quae etiam ita exprimi potest

$$\left(\frac{\frac{dQ}{Q} - \frac{dP}{P}}{Rdz}\right) + \left(\frac{\frac{dR}{R} - \frac{dQ}{Q}}{Pdx}\right) + \left(\frac{\frac{dP}{P} - \frac{dR}{R}}{Qdy}\right) = 0.$$

SCHOLION 1

5. Quemadmodum omnes aequationes differentiales inter binas variabiles semper sunt reales semperque per eas relatio certa inter ipsas variabiles definitur, ita hinc discimus rem secus se habere in aequationibus differentialibus, quae tres variabiles involvant, atque huiusmodi aequationes

$$Pdx + Qdy + Rdz = 0$$

non certam relationem inter ipsas quantitates finitas x, y et z declarare, nisi quantitates P, Q, R ita fuerint comparatae, ut character inventus locum habeat. Ex quo intelligitur infinitas huiusmodi aequationes differentiales inter ternas variabiles proponi posse, quibus nulla prorsus relatio finita conveniat et quae propterea nihil plane definiant. Pro arbitrio scilicet huiusmodi aequationes formari possunt nullo scopo proposito, ad quem sint accommodatae; statim enim ac certum quoddam problema ad aequationem differentialem inter ternas variabiles perducit, semper necesse est characterem assignatum ei convenire, cum alioquin nihil omnino significaret. Talis aequatio nihil significans est exempli gratia $zdx + xdy + ydz = 0$ neque pro z ulla quidem functio ipsarum x et y cogitari potest, quae isti aequationi satisfaciat; quin etiam character noster pro hoc exemplo dat $-x-y-z$, quae quantitas, cum non evanescat, absurditatem illius aequationis declarat.

SCHOLION 2

6. Quo character inventus facilius ad quosvis casus oblatos accommodari queat, ex aequatione

$$Pdx + Qdy + Rdz = 0$$

primo evolvantur sequentes valores

$$\left(\frac{dQ}{dz}\right) - \left(\frac{dR}{dy}\right) = L, \quad \left(\frac{dR}{dx}\right) - \left(\frac{dP}{dz}\right) = M, \quad \left(\frac{dP}{dy}\right) - \left(\frac{dQ}{dx}\right) = N$$

et character noster hac continebitur expressione

$$LP + MQ + NR,$$

quae si evanescat, aequatio proposita erit realis et aequationem quandam finitam agnoscet; sin autem ea ad nihilum non redigatur, aequatio proposita erit absurda atque de eius integratione ne cogitandum quidem erit.

Ita in exemplo supra posito erit

$$P = z, \quad Q = x, \quad R = y,$$

hinc

$$L = -1, \quad M = -1 \quad \text{et} \quad N = -1,$$

unde character $-x-y-z$ absurditatem indicat. Proferamus vero etiam exemplum aequationis realis

$$dx(yy + nyz + zz) - x(y + nz)dy - xzdz = 0,$$

in qua ob

$$P = yy + nyz + zz, \quad Q = -xy - nxz \quad \text{et} \quad R = -xz$$

erit

$$L = -nx, \quad M = -3z - ny \quad \text{et} \quad N = 3y + 2nz,$$

unde

$$LP + MQ + NR = -nx(yy + nyz + zz) + x(y + nz)(3z + ny) - xz(3y + 2nz)$$
$$= x(-nyy - nnyz - nzz + 3yz + 3nzz + nyy + nnyz - 3yz - 2nzz) = 0,$$

quare, cum hic character evanescat, aequatio haec differentialis pro reali est habenda. Simili modo proposita hac aequatione

$$2dx(y + z) + dy(x + 3y + 2z) + dz(x + y) = 0$$

ob

$$P = 2y + 2z, \quad Q = x + 3y + 2z, \quad R = x + y$$

fit

$$L = 2 - 1 = 1, \quad M = 1 - 2 = -1 \quad \text{et} \quad N = 2 - 1 = 1$$

hincque

$$LP + MQ + NR = 2y + 2z - x - 3y - 2z + x + y = 0,$$

unde ista aequatio differentialis erit realis.

PROBLEMA 2

7. *Proposita aequatione differentiali inter ternas variabiles x, y, z, quae sit realis, eius integrale investigare, unde pateat, qualis functio una earum sit binarum reliquarum.*

SOLUTIO

Sit aequatio differentialis proposita

$$Pdx + Qdy + Rdz = 0,$$

in qua P, Q, R eiusmodi sint functiones ipsarum x, y et z, ut character realitatis ante inventus satisfaciat. Nisi enim ista aequatio esset realis, ridiculum foret eius integrationem tentare. Sumamus ergo hanc aequationem esse realem atque dabitur relatio inter ipsas quantitates x, y et z aequationi propositae satisfaciens; ad quam inveniendam perpendatur, si in aequatione

integrali una variabilium, puta z, constans spectetur, ex eius differentiali nihilo aequali posito nasci debere aequationem

$$Pdx + Qdy = 0.$$

Vicissim ergo una variabili, puta z, ut constante tractata integratio aequationis differentialis $Pdx + Qdy = 0$, quae duas tantum variabiles continet, perducet ad aequationem integralem quaesitam, si modo in quantitatem constantem per integrationem ingressam illa quantitas z rite involvatur. Ex quo hanc regulam pro integratione aequationis propositae colligimus.

Consideretur una variabilium, puta z, ut constans, ut habeatur haec aequatio $Pdx + Qdy = 0$ duas tantum variabiles x et y implicans; tum eius investigetur aequatio integralis completa, quae ergo constantem arbitrariam C complectetur. Deinde haec constans C consideretur ut functio quaecunque ipsius z atque hac z nunc etiam pro variabili habita aequatio integralis inventa denuo differentietur, ut omnes tres x, y et z tanquam variabiles tractentur, et aequatio differentialis resultans comparetur cum proposita $Pdx + Qdy + Rdz = 0$, ubi quidem functiones P et Q sponte prodibunt, at functio R cum ea quantitate, qua elementum dz afficitur, collata determinabit rationem, qua quantitas z in illam litteram C ingreditur, sicque obtinebitur aequatio integralis quaesita, quae simul erit completa, cum semper in illa litterae C pars quaedam constans vere arbitraria relinquatur, cum haec determinatio ex differentiali ipsius C sit petenda.

COROLLARIUM 1

8. Reducitur ergo integratio huiusmodi aequationum differentialium tres variabiles continentium ad integrationem aequationum differentialium inter duas tantum variabiles, quae ergo, quoties licet, per methodos in superiori libro traditas est instituenda.

COROLLARIUM 2

9. Haec ergo integratio tribus modis institui potest, prout primo vel z vel y vel x tanquam constans spectatur. Semper autem necesse est, ut eadem aequatio integralis resultet, siquidem aequatio differentialis fuerit realis.

COROLLARIUM 3

10. Quodsi haec methodus tentetur in aequatione differentiali impossibili, determinatio illius constantis C non ita succedet, ut eam variabilem, quae pro constante est habita, solam involvat; atque etiam ex hoc criterium realitatis peti poterit.

SCHOLION

11. Quo haec operatio facilius intelligatur, periculum faciamus primo in aequatione impossibili hac

$$zdx + xdy + ydz = 0.$$

Hic sumta z pro constante erit

$$zdx + xdy = 0 \quad \text{seu} \quad \frac{zdx}{x} + dy = 0,$$

cuius integrale est $zlx + y = C$ existente C functione ipsius z. Differentietur ergo haec aequatio sumendo etiam z variabile positoque $dC = Ddz$, ut D sit etiam functio ipsius z tantum, erit

$$\frac{zdx}{x} + dy + dzlx = Ddz \quad \text{seu} \quad zdx + xdy + dz(xlx - Dx) = 0;$$

deberet ergo esse $xlx - Dx = y$ seu $D = lx - \frac{y}{x}$, quod est absurdum.

Deinde in aequatione reali

$$2dx(y + z) + dy(x + 3y + 2z) + dz(x + y) = 0$$

operatio exposita ita instituatur. Sumatur y constans, ut sit

$$2dx(y + z) + dz(x + y) = 0 \quad \text{seu} \quad \frac{2dx}{x + y} + \frac{dz}{y + z} = 0,$$

cuius integrale est

$$2l(x + y) + l(y + z) = C,$$

ubi C etiam y involvat. Sit ergo $dC = Ddy$ et sumto etiam y variabili differentiatio praebet

$$\frac{2dx + 2dy}{x + y} + \frac{dy + dz}{y + z} = Ddy$$

seu

$$2dx(y + z) + 2dy(y + z) + dy(x + y) + dz(x + y) = Ddy(x + y)(y + z),$$

quae expressio cum forma proposita collata praebet $D = 0$, ideoque $dC = 0$ et C fit constans vera, ita ut integrale sit

$$(x + y)^2(y + z) = \text{Const.}$$

Huiusmodi igitur exempla aliquot evolvamus.

EXEMPLUM 1

12. *Huius aequationis differentialis realis*

$$dx(y + z) + dy(x + z) + dz(x + y) = 0$$

integrale investigare.

Primo quidem patet hanc aequationem esse realem, cum sit

$$P = y + z, \quad Q = x + z, \quad R = x + y,$$
$$L = 1 - 1 = 0, \quad M = 1 - 1 = 0, \quad N = 1 - 1 = 0.$$

Sumatur igitur z constans et aequatio prodibit

$$dx(y + z) + dy(x + z) = 0 \quad \text{seu} \quad \frac{dx}{x + z} + \frac{dy}{y + z} = 0,$$

cuius integrale est

$$l(x + z) + l(y + z) = f:z;$$

statuatur ergo

$$(x + z)(y + z) = Z,$$

ubi natura functionis Z ex differentiatione debet erui. Fit autem

$$dx(y + z) + dy(x + z) + dz(x + y + 2z) = dZ,$$

a qua si proposita auferatur, relinquitur $2zdz = dZ$, hinc $Z = zz + C$, ita ut aequatio integralis completa sit

$$(x + z)(y + z) = zz + C \quad \text{seu} \quad xy + xz + yz = C,$$

quae quidem ex ipsa proposita

$$ydx + zdx + xdy + zdy + xdz + ydz = 0$$

facile elicitur, cum bina membra iuncta sint integrabilia.

EXEMPLUM 2

13. *Huius differentialis aequationis realis*

$$dx(ay-bz)+dy(cz-ax)+dz(bx-cy)=0$$

aequationem integralem completam invenire.

Realitas huius aequationis ita ostenditur. Cum sit

$$P=ay-bz,\quad Q=cz-ax,\quad R=bx-cy,$$

erit

$$L=2c,\quad M=2b,\quad N=2a$$

hincque manifesto $LP+MQ+NR=0$.

Iam sumatur z constans, ut habeatur

$$\frac{dx}{cz-ax}+\frac{dy}{ay-bz}=0,\quad \text{ergo}\quad \frac{1}{a}l\frac{ay-bz}{cz-ax}=f:z;$$

statuatur ergo

$$\frac{ay-bz}{cz-ax}=Z$$

et differentiatio praebet

$$\frac{adx(ay-bz)+ady(cz-ax)+adz(bx-cy)}{(cz-ax)^2}=dZ,$$

ex cuius comparatione cum proposita fit $dZ=0$ et $Z=C$, ita ut aequatio integralis completa sit

$$\frac{ay-bz}{cz-ax}=n\quad \text{seu}\quad ay+nax=(b+nc)z.$$

Quodsi aequatio integralis ponatur

$$Ax+By+Cz=0,$$

hae constantes ita debent esse comparatae, ut sit

$$Ac+Bb+Ca=0,$$

sicque constans arbitraria concinnius inducitur.

COROLLARIUM

14. Haec ergo aequatio integrabilis redditur, si dividatur per $(cz-ax)^2$, atque ob eandem rationem etiam hi divisores $(ay-bz)^2$ et $(bx-cy)^2$ idem praestant. Vi enim integralis hi divisores constantem inter se tenent rationem. Namque si $\frac{ay-bz}{cz-ax}=n$, erit

$$\frac{bx-cy}{cz-ax}=\frac{-b-nc}{a} \quad \text{et} \quad \frac{bx-cy}{ay-bz}=\frac{-b-nc}{na}.$$

EXEMPLUM 3

15. *Huius aequationis differentialis realis*

$$dx(yy+yz+zz)+dy(zz+xz+xx)+dz(xx+xy+yy)=0$$

aequationem integralem completam investigare.

Realitas huius aequationis inde patet, quod sit

$$P=yy+yz+zz,\quad Q=zz+xz+xx,\quad R=xx+xy+yy$$

hincque

$$L=2z+x-x-2y=2(z-y),\quad M=2x+y-y-2z=2(x-z),$$
$$N=2y+z-z-2x=2(y-x),$$

unde fit

$$LP+MQ+NR=2(z^3-y^3)+2(x^3-z^3)+2(y^3-x^3)=0.$$

Ad integrale ergo investigandum sumatur z constans eritque

$$\frac{dx}{xx+xz+zz}+\frac{dy}{yy+yz+zz}=0,$$

cuius integrale est

$$\frac{2}{z\sqrt{3}}\,\text{Ang. tang.}\,\frac{x\sqrt{3}}{2z+x}+\frac{2}{z\sqrt{3}}\,\text{Ang. tang.}\,\frac{y\sqrt{3}}{2z+y}=f:z,$$

quae per collectionem horum angulorum abit in

$$\frac{2}{z\sqrt{3}}\,\text{Ang. tang.}\,\frac{(xz+yz+xy)\sqrt{3}}{2zz+xz+yz-xy}=f:z.$$

Statuatur ergo

$$\frac{xz+yz+xy}{2zz+xz+yz-xy}=Z$$

haecque aequatio differentietur sumtis omnibus tribus x, y et z variabilibus ac prodibit

$$\frac{2zdx(yy+yz+zz)+2zdy(zz+xz+xx)-2xdz(zz+yz+yy)-2ydz(zz+xz+xx)}{(2zz+xz+yz-xy)^2}=dZ;$$

cum igitur ex aequatione proposita sit

$$dx(yy+yz+zz)+dy(zz+xz+xx)=-dz(xx+xy+yy),$$

erit facta substitutione

$$\frac{-2zdz(xx+xy+yy)-2xdz(zz+yz+yy)-2ydz(zz+xz+xx)}{(2zz+xz+yz-xy)^2}=dZ$$

seu

$$\frac{-2dz(xxz+xzz+yyz+yzz+xxy+xyy+3xyz)}{(2zz+xz+yz-xy)^2}=dZ,$$

quae in hanc formam reducitur

$$\frac{-2dz(x+y+z)(xy+xz+yz)}{(2zz+xz+yz-xy)^2}=dZ.$$

At ob $Z=\frac{xy+xz+yz}{2zz+xz+yz-xy}$ erit

$$\frac{-2ZZdz(x+y+z)}{xy+xz+yz}=dZ \quad \text{seu} \quad \frac{-dZ}{ZZ}=\frac{2dz(x+y+z)}{xy+xz+yz}.$$

Necesse ergo est, ut etiam $\frac{xy+xz+yz}{x+y+z}$ sit functio ipsius z tantum, quae vocetur Σ, ut sit

$$-\frac{dZ}{ZZ}=\frac{2dz}{\Sigma}.$$

Verum ex sola forma functionis Z negotium confici oportet, quod ita expediri potest. Cum sit $Z=\frac{xy+xz+yz}{2zz+xz+yz-xy}$, erit

$$1+Z=\frac{2zz+2xz+2yz}{2zz+xz+yz-xy}, \quad \text{hinc} \quad \frac{1+Z}{Z}=\frac{2z(x+y+z)}{xy+xz+yz},$$

cuius valoris ope quantitates x et y ex aequatione differentiali eliduntur, fitque

$$-\frac{dZ}{ZZ}=dz\frac{2(x+y+z)}{xy+xz+yz}=dz\frac{1+Z}{Zz},$$

unde

$$\frac{-dZ}{Z(1+Z)}=\frac{dz}{z}=\frac{-dZ}{Z}+\frac{dZ}{1+Z}$$

et integrando $lz=l\frac{1+Z}{Z}+la$. Ergo

$$\frac{1+Z}{Z}=\frac{z}{a} \quad \text{et} \quad Z=\frac{a}{z-a},$$

ita ut aequatio integralis quaesita sit

$$\frac{a}{z-a}=\frac{xy+xz+yz}{2zz+xz+yz-xy} \quad \text{seu} \quad xy+xz+yz=a(x+y+z),$$

quae simplicissima forma statim colligitur ex aequatione

$$\frac{2z(x+y+z)}{xy+xz+yz}=\frac{1+Z}{Z}=\frac{z}{a}.$$

COROLLARIUM

15[a][1]. Cum aequationis propositae integrale completum sit

$$xy+xz+yz=a(x+y+z) \quad \text{seu} \quad \frac{xy+xz+yz}{x+y+z}=\text{Const.},$$

ex huius differentiatione etiam ipsa aequatio proposita resultare deprehenditur. Unde patet aequationem propositam integrabilem reddi, si dividatur per $(x+y+z)^2$ vel etiam per $(xy+xz+yz)^2$.

SCHOLION

16. Ex hoc exemplo intelligitur determinationem functionis per integrationem illatae interdum haud exiguis difficultatibus esse obnoxiam, siquidem hic functionem Z non sine ambagibus elicuimus. Verum et hic ista investigatio multo facilius institui potuisset; statim enim atque invenimus

$$\frac{xy+xz+yz}{2zz+xz+yz-xy}=Z=f:z,$$

1) In editione principe falso numerus 15 iteratur. F. E.

hanc ipsam expressionem concinniorem reddere licuisset. Nempe cum sit

$$\frac{1}{Z}=\frac{2zz+xz+yz-xy}{xy+xz+yz},$$

erit

$$1+\frac{1}{Z}=\frac{2z(x+y+z)}{xy+xz+yz}$$

ideoque

$$\frac{xy+xz+yz}{x+y+z}=\frac{2Zz}{1+Z}=f:z.$$

Relicta ergo functione Z statim ponatur

$$\frac{xy+xz+yz}{x+y+z}=\Sigma=f:z$$

et sumtis differentialibus per se liquebit fieri $d\Sigma=0$ ideoque $\Sigma=$ Const.

Adhuc facilius hoc problema resolvitur, si etiam sumto y constante eius integrale quaeratur; tum enim simili modo pervenitur ad huiusmodi aequationem

$$\frac{xy+xz+yz}{x+y+z}=Y=f:y;$$

quare cum haec expressio aeque esse debeat functio ipsius z atque ipsius y, necesse est, ut ea sit constans, eritque propterea aequatio integralis completa

$$xy+xz+yz=a(x+y+z).$$

EXEMPLUM 4

17. *Huius aequationis differentialis realis*

$$dx(xx-yy+zz)-zzdy+zdz(y-x)+\frac{xdz}{z}(yy-xx)=0$$

aequationem integralem completam investigare.

Realitas huius aequationis ita ostenditur. Ob

$$P=xx-yy+zz,\quad Q=-zz,\quad R=z(y-x)+\frac{x}{z}(yy-xx)$$

erit

$$L=-3z-\frac{2xy}{z},\quad M=-3z+\frac{yy}{z}-\frac{3xx}{z},\quad N=-2y,$$

unde calculo subducto formula $LP+MQ+NR$ evanescit.

Sumamus iam z constans et habebimus hanc aequationem

$$dx(xx - yy + zz) - zzdy = 0,$$

cuius quidem integratio non constaret, nisi perspiceremus ei satisfacere particulariter $y = x$. Hinc autem ponendo $y = x + \frac{zz}{v}$ integrale completum eruere poterimus; fit enim

$$dx\left(zz - \frac{2xzz}{v} - \frac{z^4}{vv}\right) - zzdx + \frac{z^4 dv}{vv} = 0$$

hincque

$$dv - \frac{2xv\,dx}{zz} = dx,$$

quae per $e^{\frac{-xx}{zz}}$ multiplicata praebet integrale

$$e^{\frac{-xx}{zz}} v = \int e^{\frac{-xx}{zz}} dx + f:z,$$

ubi quidem notandum est in integratione formulae $\int e^{\frac{-xx}{zz}} dx$ quantitatem z ut constantem tractari esseque $v = \frac{zz}{y-x}$, ita ut sit

$$\int e^{\frac{-xx}{zz}} dx = \frac{e^{\frac{-xx}{zz}} zz}{y-x} + Z.$$

Quodsi iam hanc aequationem differentiare velimus sumta etiam z variabili, difficultas hic occurrit, quomodo quantitatis $\int e^{\frac{-xx}{zz}} dx$ differentiale ex variabilitate ipsius z oriundum definiri debeat. Hic ex principiis repeti debet, si fuerit $dV = Sdx + Tdz$, fore $\left(\frac{dT}{dx}\right) = \left(\frac{dS}{dz}\right)$ ideoque, si z constans sumatur, $T = \int dx \left(\frac{dS}{dz}\right)$. Iam nostro casu est

$$S = e^{\frac{-xx}{zz}} \quad \text{et} \quad V = \int e^{\frac{-xx}{zz}} dx$$

sumta z constante; quare cum sit $\left(\frac{dS}{dz}\right) = e^{\frac{-xx}{zz}} \frac{2xx}{z^3}$, ergo

$$T = \frac{2}{z^3} \int e^{\frac{-xx}{zz}} xxdx.$$

Quocirca quantitatis $\int e^{\frac{-xx}{zz}} dx$ differentiale plenum ex variabilitate utriusque x

et z oriundum est

$$e^{\frac{-xx}{zz}}dx + \frac{2dz}{z^3}\int e^{\frac{-xx}{zz}}xxdx,$$

cui aequari debet alterius partis $\frac{e^{\frac{-xx}{zz}}zz}{y-x} + Z$ differentiale, quod est

$$e^{\frac{-xx}{zz}}\left(\frac{2zdz}{y-x} - \frac{zzdy - zzdx}{(y-x)^2} + \frac{2xxdz - 2xzdx}{z(y-x)}\right) + dZ.$$

Turbat vero adhuc formula integralis $\int e^{\frac{-xx}{zz}}xxdx$, in qua z pro constante habetur; reduci autem potest ad priorem $\int e^{\frac{-xx}{zz}}dx$, si ponatur

$$\int e^{\frac{-xx}{zz}}xxdx = Ae^{\frac{-xx}{zz}}x + B\int e^{\frac{-xx}{zz}}dx;$$

prodit enim sola x pro variabili habita differentiando

$$xxdx = Adx - \frac{2Axxdx}{zz} + Bdx,$$

ergo

$$A = -\frac{1}{2}zz \quad \text{et} \quad B = -A = \frac{1}{2}zz,$$

ita ut sit

$$\int e^{\frac{-xx}{zz}}xxdx = -\frac{1}{2}e^{\frac{-xx}{zz}}xzz + \frac{1}{2}zz\int e^{\frac{-xx}{zz}}dx.$$

Quare cum sit

$$\int e^{\frac{-xx}{zz}}dx = \frac{e^{\frac{-xx}{zz}}zz}{y-x} + Z,$$

erit

$$\int e^{\frac{-xx}{zz}}xxdx = -\frac{1}{2}e^{\frac{-xx}{zz}}xzz + \frac{e^{\frac{-xx}{zz}}z^4}{2(y-x)} + \frac{1}{2}Zzz.$$

Facta ergo substitutione haec orietur aequatio differentialis

$$e^{\frac{-xx}{zz}}\left(dx - \frac{xdz}{z} + \frac{zdz}{y-x}\right) + \frac{Zdz}{z}$$

$$= e^{\frac{-xx}{zz}}\left(\frac{2zdz}{y-x} - \frac{zzdy}{(y-x)^2} + \frac{zzdx}{(y-x)^2} - \frac{2xdx}{y-x} + \frac{2xxdz}{z(y-x)}\right) + dZ,$$

quae transit in hanc formam

$$e^{\frac{-xx}{zz}}\left(\frac{dx(y+x)}{y-x}-\frac{zzdx}{(y-x)^2}+\frac{zzdy}{(y-x)^2}-\frac{zdz}{y-x}-\frac{x(y+x)dz}{z(y-x)}\right)=\frac{zdZ-Zdz}{z}$$

seu

$$\frac{e^{\frac{-xx}{zz}}}{(y-x)^2}\left(dx(yy-xx-zz)+zzdy-zdz(y-x)-\frac{xdz}{z}(yy-xx)\right)=\frac{zdZ-Zdz}{z};$$

qua cum proposita collata evidens est esse debere

$$zdZ-Zdz=0 \quad \text{seu} \quad Z=nz,$$

ita ut aequationis propositae integrale completum sit

$$\int e^{\frac{-xx}{zz}}dx=\frac{e^{\frac{-xx}{zz}}zz}{y-x}+nz,$$

siquidem in integrali $\int e^{\frac{-xx}{zz}}dx$ quantitas z pro constante habeatur.

COROLLARIUM

18. Aequatio ergo proposita integrabilis redditur, si multiplicetur per[1])

$$\frac{1}{(y-x)^2}e^{\frac{-xx}{zz}};$$

ac tum integrale est ipsa aequatio, quam invenimus.

SCHOLION 1

19. Exemplum hoc imprimis est notatu dignum, quod in eius solutione quaedam artificia sunt in subsidium vocata, quibus in praecedentibus non erat opus. Per formulam autem $\int e^{\frac{-xx}{zz}}dx$ integrale non satis determinatum videtur. Cum enim in ea z constans ponatur, constans per integrationem introducenda per nz non definitur, siquidem lex non praescribitur, secundum quam integrale $\int e^{\frac{-xx}{zz}}dx$ capi oporteat, utrum ita, ut evanescat facto $x=0$,

1) In paragrapho sequenti aequationis multiplicator accuratius determinatur; expressio enim hic data per z dividenda est. F. E.

an alio quocunque modo. Dubium autem hoc diluetur, si aequationem inventam per z dividamus, ut formula integralis sit $\int e^{\frac{-xx}{zz}}\frac{dx}{z}$; ubi cum $\frac{dx}{z}$ sit $d.\frac{x}{z}$, evidens est ea exprimi functionem quandam ipsius $\frac{x}{z}$, ac si ponatur $\frac{x}{z}=p$, fore aequationem nostram integralem

$$\int e^{-pp}dp + \text{Const.} = e^{-pp}.\frac{z}{y-x};$$

neque hic amplius conditio illa, qua in formula integrali quantitas z pro constante sit habenda, locum habet, sed integrale perinde determinatur, ac si aequatio duas tantum variabiles contineret. Hanc circumstantiam si perpendissemus, plenum differentiale formulae $\int e^{\frac{-xx}{zz}}dx$ ex variabilitate utriusque x et z nullam difficultatem peperisset. Postquam enim pervenimus ad aequationem

$$\int e^{\frac{-xx}{zz}}dx = e^{\frac{-xx}{zz}}.\frac{zz}{y-x}+f:z,$$

eam ita repraesentemus

$$\int e^{\frac{-xx}{zz}}\frac{dx}{z} = \int e^{\frac{-xx}{zz}}d.\frac{x}{z} = e^{\frac{-xx}{zz}}\frac{z}{y-x}+Z;$$

ubi cum in formulam integralem etiam variabilitas ipsius z sit inducta, si ea differentietur sumtis omnibus x, y et z variabilibus, orietur

$$e^{\frac{-xx}{zz}}\left(\frac{dx}{z}-\frac{xdz}{zz}\right) = e^{\frac{-xx}{zz}}\left(\frac{dz}{y-x}+\frac{zdx-zdy}{(y-x)^2}-\frac{2xdx}{z(y-x)}+\frac{2xxdz}{zz(y-x)}\right)+dZ$$

seu

$$e^{\frac{-xx}{zz}}\left(\frac{dx(y+x)}{z(y-x)}-\frac{zdx}{(y-x)^2}+\frac{zdy}{(y-x)^2}-\frac{xdz(y+x)}{zz(y-x)}-\frac{dz}{y-x}\right)=dZ,$$

quae reducitur ad hanc formam

$$\frac{e^{\frac{-xx}{zz}}}{z(y-x)^2}\left(dx(yy-xx-zz)+zzdy-zdz(y-x)-\frac{xdz}{z}(yy-xx)\right)=dZ,$$

unde patet esse debere $dZ=0$ et $Z=\text{Const.}$, sicque elicitur aequatio integralis ante inventa.

SCHOLION 2

20. Idem integrale prodiisset, si loco z altera reliquarum x vel y pro constante fuisset assumta; ubi in genere notari convenit, si huiusmodi aequationem

$$Pdx + Qdy + Rdz = 0$$

sumta z constante tractare licuerit, etiam resolutionem, quaecunque trium variabilium pro constante assumatur, succedere debere, etiamsi id quandoque minus perspiciatur. Ita in aequatione proposita si y pro constante habeatur, resolvenda erit haec aequatio

$$dx(xx + zz - yy) - zdz(x - y) - \frac{xdz}{z}(xx - yy) = 0;$$

quae per z multiplicata cum in hanc formam abeat

$$(zdx - xdz)(xx + zz - yy) + yzzdz = 0,$$

facile patet eam simpliciorem reddi ponendo $x = pz$; tum enim ob

$$zdx - xdz = zzdp$$

prodibit

$$dp(ppzz + zz - yy) + ydz = 0.$$

Sit porro $z = qy$ fietque

$$dp(ppqq + qq - 1) + dq = 0,$$

cui cum satisfaciat $q = \frac{1}{p}$, statuatur $q = \frac{1}{p} + \frac{1}{r}$ habebiturque

$$dp\left(\frac{2p}{r} + \frac{pp}{rr} + \frac{1}{pp} + \frac{2}{pr} + \frac{1}{rr}\right) - \frac{dp}{pp} - \frac{dr}{rr} = 0$$

seu

$$dp(2ppr + p^3 + 2r + p) - pdr = 0 \quad \text{vel} \quad dr - \frac{2rdp(pp+1)}{p} = dp(pp+1),$$

quae multiplicata per $\frac{1}{pp}e^{-pp}$ et integrata dat

$$e^{-pp}\frac{r}{pp} = \int e^{-pp}\frac{dp(1+pp)}{pp}.$$

At
$$\int e^{-pp}\frac{dp}{pp} = -e^{-pp}\frac{1}{p} - 2\int e^{-pp}dp,$$
unde
$$e^{-pp}\left(\frac{r}{pp}+\frac{1}{p}\right) = -\int e^{-pp}dp.$$
Cum nunc sit $p=\frac{x}{z}$ et $\frac{1}{r}=\frac{z}{y}-\frac{z}{x}=\frac{z(x-y)}{xy}$, erit
$$r=\frac{xy}{z(x-y)},\quad \frac{r}{pp}=\frac{yz}{x(x-y)}\quad \text{et}\quad \frac{r}{pp}+\frac{1}{p}=\frac{z}{x-y}.$$
Unde aequatio nostra integralis erit
$$\int e^{\frac{-xx}{zz}}d.\frac{x}{z} = e^{\frac{-xx}{zz}}\frac{z}{y-x}+f:y,$$
cuius differentiale, si etiam y pro variabili habeatur, cum aequatione proposita comparatum dabit ut ante $f:y=\text{Const.}$

Caeterum cum in his exemplis variabiles x, y, z ubique eundem dimensionum numerum impleant, methodum generalem huiusmodi aequationes tractandi exponam.

PROBLEMA 3

21. *Si in aequatione differentiali*
$$Pdx+Qdy+Rdz=0$$
functiones P, Q, R fuerint homogeneae ipsarum x, y et z eiusdem numeri dimensionum, eius integrationem, siquidem fuerit realis, investigare.

SOLUTIO

Sit n numerus dimensionum, quas ternae variabiles x, y et z in functionibus P, Q, R constituunt, ac posito $x=pz$ et $y=qz$ fiet
$$P=z^nS,\quad Q=z^nT\quad \text{et}\quad R=z^nV,$$
ita ut iam S, T, V futurae sint functiones binarum tantum variabilium p et q. Cum iam sit
$$dx=pdz+zdp\quad \text{et}\quad dy=qdz+zdq,$$

aequatio nostra hanc induet formam

$$dz(pS+qT+V)+Szdp+Tzdq=0 \quad \text{seu} \quad \frac{dz}{z}+\frac{Sdp+Tdq}{pS+qT+V}=0,$$

quae aequatio realis esse nequit, nisi formula differentialis binas variabiles p et q involvens $\frac{Sdp+Tdq}{pS+qT+V}$ per se fuerit integrabilis; quod eveniet, si fuerit

$$(qT+V)\left(\frac{dS}{dq}\right)+pT\left(\frac{dS}{dp}\right)-(pS+V)\left(\frac{dT}{dp}\right)-qS\left(\frac{dT}{dq}\right)-S\left(\frac{dV}{dq}\right)+T\left(\frac{dV}{dp}\right)=0.$$

Quoties ergo hic character locum habet, nostra aequatio erit realis eiusque integrale erit

$$lz+\int\frac{Sdp+Tdq}{pS+qT+V}=\text{Const.},$$

ubi tantum opus est, ut loco litterarum p et q valores assumti $\frac{x}{z}$ et $\frac{y}{z}$ restituantur.

COROLLARIUM 1

22. Ita in nostro primo exemplo (§ 12) cum sit

$$P=y+z, \quad Q=x+z, \quad R=x+y,$$

erit

$$S=q+1, \quad T=p+1, \quad V=p+q$$

et

$$\frac{dz}{z}+\frac{(q+1)dp+(p+1)dq}{2pq+2p+2q}=0,$$

cuius integrale est

$$lz+\frac{1}{2}l(pq+p+q)=\frac{1}{2}l(xy+xz+yz)=C$$

seu

$$xy+xz+yz=C.$$

COROLLARIUM 2

23. In secundo exemplo (§ 13) est

$$P=ay-bz, \quad Q=cz-ax, \quad R=bx-cy,$$

hinc

$$S=aq-b, \quad T=c-ap, \quad V=bp-cq.$$

Ergo

$$\frac{dz}{z} + \frac{(aq-b)\,dp + (c-ap)\,dq}{0} = 0$$

hincque

$$(aq-b)dp + (c-ap)dq = 0$$

et integrando

$$l\,\frac{aq-b}{c-ap} = l\,\frac{ay-bz}{cz-ax} = C.$$

COROLLARIUM 3

24. In tertio exemplo (§ 15) fit

$$S = qq + q + 1, \quad T = pp + p + 1 \quad \text{et} \quad V = pp + pq + qq$$

hincque

$$\frac{dz}{z} + \frac{dp(qq+q+1) + dq(pp+p+1)}{ppq+pqq+pp+3pq+qq+p+q} = 0,$$

qui denominator est $= (p+q+1)(pq+p+q)$, unde haec fractio resolvitur in has duas

$$\frac{-dp-dq}{p+q+1} + \frac{dp(q+1)+dq(p+1)}{pq+p+q},$$

ex quo integrale a logarithmis ad numeros perductum oritur

$$\frac{z(pq+p+q)}{p+q+1} = \frac{xy+xz+yz}{x+y+z} = C.$$

COROLLARIUM 4

25. In exemplo quarto (§ 17) fit

$$S = pp - qq + 1, \quad T = -1, \quad V = q - p + p(qq - pp)$$

hincque

$$\frac{dz}{z} + \frac{dp\,(pp-qq+1) - dq}{0} = 0$$

ideoque

$$dq = dp(pp - qq + 1).$$

Cum ergo satisfaciat $q = p$, ponatur $q = p + \frac{1}{r}$; fiet $dr - 2prdp = dp$ et integrando

$$e^{-pp}r = \int e^{-pp}dp = e^{-pp}\frac{1}{q-p},$$

ita ut integrale sit

$$e^{\frac{-xx}{zz}}\frac{z}{y-x}=\int e^{\frac{-xx}{zz}}d.\frac{x}{z}+\text{Const.}$$

SCHOLION

26. Cum igitur aequationes differentiales tres variabiles involventes nullam habeant difficultatem sibi propriam, quoniam earum resolutio, siquidem fuerint reales, semper ad aequationes differentiales duarum variabilium reduci potest, hoc argumentum fusius non prosequor. Quod enim ad eiusmodi aequationes differentiales trium variabilium attinet, in quibus ipsa differentialia ad plures dimensiones ascendunt, veluti est

$$Pdx^2+Qdy^2+Rdz^2+2Sdxdy+2Tdxdz+2Vdydz=0,$$

de iis generatim tenendum est, nisi per radicis extractionem ad formam

$$Pdx+Qdy+Rdz=0$$

reduci queant, eas semper esse absurdas. Quomodocunque enim aequatio integralis esset comparata, ex ea valor ipsius z ita definiri posset, ut z aequetur functioni binarum variabilium x et y, unde foret $dz=pdx+qdy$, neque hae variabiles x et y ullo modo a se penderent. Hic ergo valor $pdx+qdy$ loco dz in aequatione differentiali substitutus ita satisfacere deberet, ut omnes termini se mutuo destruerent, quod autem fieri non posset, si ex aequationis resolutione dz ita definiretur, ut differentialia dx et dy signis radicalibus essent involuta. Hinc aequatio illa exempli loco allata, cum per resolutionem det

$$dz=\frac{-Tdx-Vdy\pm\sqrt{((TT-PR)dx^2+2(TV-RS)dxdy+(VV-QR)dy^2)}}{R},$$

realis esse nequit, nisi radix extrahi queat, hoc est, nisi ipsa aequatio in factores formae

$$Pdx+Qdy+Rdz$$

resolvi possit. Atque etiamsi hoc eveniat et hi factores nihilo aequales statuantur, tamen aequatio non erit realis, nisi criterium supra traditum locum habeat.

Ex his perspicuum est ne eiusmodi quidem aequationes, quae quatuor pluresve variabiles involvant, plus difficultatis habere.

PROBLEMA 4

27. *Si V sit functio quaecunque binarum variabilium x et y, in formula autem integrali $\int V dx$ quantitas y pro constante sit habita, definire huius formae $\int V dx$ differentiale, si praeter x etiam y variabilis assumatur.*

SOLUTIO

Ponatur ista formula integralis

$$\int V dx = Z$$

eritque Z utique functio ambarum variabilium x et y, etiamsi in ipsa integratione y pro constante habeatur. Evidens autem est, si vicissim in differentiatione y constans sumatur, fore $dZ = V dx$. Quare si etiam y variabilis statuatur, differentiale ipsius $Z = \int V dx$ huiusmodi habebit formam

$$dZ = V dx + Q dy$$

et quaestio huc redit, ut ista quantitas Q determinetur. Quia autem forma $V dx + Q dy$ est verum differentiale, necesse est sit $\left(\frac{dV}{dy}\right) = \left(\frac{dQ}{dx}\right)$ hincque $dx\left(\frac{dQ}{dx}\right) = dx\left(\frac{dV}{dy}\right)$. At $dx\left(\frac{dQ}{dx}\right)$ est differentiale ipsius Q, si y pro constante habeatur; unde Q reperietur, si formula $dx\left(\frac{dV}{dy}\right)$ ita integretur, ut y tanquam constans tractetur, seu erit

$$Q = \int dx\left(\frac{dV}{dy}\right).$$

Quocirca formulae $Z = \int V dx$ differentiale ex variabilitate utriusque x et y oriundum erit

$$dZ = V dx + dy \int dx\left(\frac{dV}{dy}\right).$$

COROLLARIUM 1

28. Quoniam V est functio ipsarum x et y, si ponatur $dV = R dx + S dy$, erit $S = \left(\frac{dV}{dy}\right)$, unde fit

$$dZ = d.\int V dx = V dx + dy \int S dx,$$

scilicet in formulae $\int S dx$ integratione, perinde ac formulae $\int V dx$, sola quantitas x pro variabili est habenda.

COROLLARIUM 2

29. Si V fuerit functio homogenea ipsarum x et y existente numero dimensionum $=n$, posito $dV = Rdx + Sdy$ erit $Rx + Sy = nV$ ideoque $S = \frac{nV}{y} - \frac{Rx}{y}$, hinc

$$\int Sdx = \frac{n}{y}\int Vdx - \frac{1}{y}\int Rxdx.$$

At ob y constans est $Rdx = dV$, hinc

$$\int Rxdx = \int xdV = Vx - \int Vdx \quad \text{ideoque} \quad \int Sdx = \frac{n+1}{y}\int Vdx - \frac{Vx}{y}$$

et

$$dZ = d.\int Vdx = Vdx - \frac{Vxdy}{y} + \frac{(n+1)dy}{y}\int Vdx.$$

COROLLARIUM 3

30. Idem facilius invenitur ex consideratione, quod functio $Z = \int Vdx$ futura sit homogenea $n+1$ dimensionum, quare posito $dZ = Vdx + Qdy$ erit $Vx + Qy = (n+1)Z$ ideoque $Q = \frac{(n+1)Z}{y} - \frac{Vx}{y}$ ut ante.

SCHOLION

31. Problemate iam ante et in praecedente quidem libro[1]) sum usus neque tamen abs re fore putavi, si id data opera hic tractarem, quandoquidem hic liber in functionibus binarum pluriumve variabilium occupatur.

Praecipuum autem negotium non in eiusmodi aequationibus differentialibus, quales in hoc capite integrare docui, versatur, quod quidem brevi esset absolutum, sed cum differentiatio functionis binarum variabilium x et y duplices formulas $\left(\frac{dV}{dx}\right)$ et $\left(\frac{dV}{dy}\right)$ suppeditet existente V huiusmodi functione, hoc loco eiusmodi quaestiones potissimum contemplabimur, quibus talis functio V ex data quacunque relatione harum duarum formularum $\left(\frac{dV}{dx}\right)$ et $\left(\frac{dV}{dy}\right)$ est definienda. Relatio autem haec per aequationem inter istas formulas et binas variabiles x et y, quam etiam ipsa functio quaesita V ingredi potest,

1) Vide *Institutionum calculi integralis* vol. I § 457 et 458, LEONHARDI EULERI *Opera omnia*, series I, vol. 11, p. 288—290. Cf. etiam *Mechanicae* tom. II § 106, LEONHARDI EULERI *Opera omnia*, series II, vol. 2, p. 44. F. E.

exprimitur, ex cuius aequationis indole divisio tractationis erit petenda. Problema scilicet generale, in quo solvendo ista sectio est occupata, ita se habet, ut ea binarum variabilium x et y functio V inveniatur, quae satisfaciat aequationi cuicunque inter quantitates x, y, V, $\left(\frac{dV}{dx}\right)$ et $\left(\frac{dV}{dy}\right)$ propositae. Quodsi in hanc aequationem altera tantum binarum formularum differentialium $\left(\frac{dV}{dx}\right)$ vel $\left(\frac{dV}{dy}\right)$ ingrediatur, resolutio non est difficilis atque ad casum aequationum differentialium duas tantum variabiles involventium reducitur; quando autem ambae istae formulae in aequatione proposita insunt, quaestio multo magis est ardua ac saepenumero ne resolvi quidem potest, etiamsi resolutio aequationum differentialium duas tantum variabiles complectentium admittatur; in hoc enim negotio, quoties resolutionem ad integrationem aequationum differentialium inter duas variabiles reducere licet, problema pro resoluto erit habendum.

Cum igitur ex aequatione proposita formula $\left(\frac{dV}{dy}\right)$ aequetur functioni utcunque ex quantitatibus x, y, V et $\left(\frac{dV}{dx}\right)$ conflatae, ex indole huius functionis, prout fuerit simplicior et vel solam formulam $\left(\frac{dV}{dx}\right)$ vel praeter eam unicam ex reliquis vel etiam binas vel adeo omnes comprehendat, tractationem sequentem distribuemus. Hoc enim ordine servato facillime apparebit, quantum adhuc praestare liceat et quantum adhuc desideretur. Praeterea vero nonnulla subsidia circa transformationem binarum formularum differentialium ad alias variabiles exponenda occurrent.

DIVISIO HUIUS SECTIONIS

32. Quo partes, quas in hac sectione pertractari convenit, clarius conspectui exponantur, quoniam hae quaestiones circa functiones binarum variabilium versantur, sint x et y binae variabiles et z earum functio ex data quadam differentialium relatione definienda, ita ut aequatio finita inter x, y et z requiratur. Ponamus autem $dz = pdx + qdy$, ita ut sit recepto signandi modo $p = \left(\frac{dz}{dx}\right)$ et $q = \left(\frac{dz}{dy}\right)$ atque ideo p et q sint formulae differentiales, quae in relationem propositam ingrediantur. In genere ergo relatio ista erit aequatio quaecunque inter quantitates p, q, x, y et z proposita atque haec sectio perfecte absolveretur, si methodus constaret ex data aequatione quacunque inter has quantitates p, q, x, y et z eruendi aequationem inter x, y et z; quod autem cum in genere ne pro functionibus quidem unicae variabilis praestari possit, multo minus hic est expectandum; ex quo eos casus tantum evolvi conveniet, qui resolutionem admittant.

Primo autem resolutio succedit, si in aequatione proposita altera formularum differentialium p vel q plane desit, ita ut aequatio vel inter p, x, y et z vel inter q, x, y et z proponatur.

Deinde aequationes, quae solas binas formulas differentiales p et q continent, ita ut altera debeat esse functio quaecunque alterius, commode resolvere licet.

Tum igitur sequentur aequationes, quae praeter p et q unicam quantitatum finitarum x vel y vel z complectantur, ex quo genere cuiusmodi casus resolvi queant, videamus.

Ordo porro postulat, ut ad aequationes, quae praeter binas formulas differentiales p et q insuper binas quantitatum finitarum vel x et y vel x et z vel y et z involvunt, progrediamur; ac denique de resolutione aequationum omnes litteras p, q, x, y et z implicantium agemus subsidia transformationis deinceps expedituri.

CAPUT II

DE RESOLUTIONE AEQUATIONUM QUIBUS ALTERA FORMULA DIFFERENTIALIS PER QUANTITATES FINITAS UTCUNQUE DATUR

PROBLEMA 4[a][1])

33. *Investigare indolem functionis z binarum variabilium x et y, ut formula differentialis $\left(\frac{dz}{dx}\right) = p$ sit quantitas constans $= a$.*

SOLUTIO

Posito ergo $dz = pdx + qdy$ ea functionis z indoles quaeritur, ut sit $p = a$ seu $dz = adx + qdy$; ad quam inveniendam sumatur y pro constante; erit $dz = adx$ et integrando $z = ax + \text{Const.}$, ubi notari oportet hanc constantem utcunque involvere posse quantitatem y. Quare ut solutionem generalem exhibeamus, erit $z = ax + f:y$ denotante $f:y$ functionem quamcunque ipsius y, quae per se nullo modo determinatur, sed penitus ab arbitrio nostro pendet. Quod etiam differentiatio vicissim declarat; si enim huius functionis $f:y$ differentiale per $dyf':y$ indicemus, erit utique $dz = adx + dyf':y$ ideoque $\left(\frac{dz}{dx}\right) = a$, prorsus uti quaestio postulat; unde patet hoc casu alteram formulam differentialem $q = \left(\frac{dz}{dy}\right)$ functioni solius y aequari, cum sit $q = f':y$[2]).

COROLLARIUM 1

34. Si ergo eiusmodi quaeratur functio z binarum variabilium x et y, ut sit $\left(\frac{dz}{dx}\right) = a$, erit $z = ax + f:y$ et altera formula differentialis $\left(\frac{dz}{dy}\right)$ necessario aequatur functioni ipsius y tantum.

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1) In editione principe falso numerus 4 iteratur. F. E.

2) Editio princeps: ... *cum sit* $q = \left(\frac{dz}{dy}\right)$. Correxit F. E.

COROLLARIUM 2

35. Si talis requiratur functio, ut sit $\left(\frac{dz}{dx}\right)=0$, ea necessario erit functio ipsius y tantum seu quantitatem x plane non involvet; cum enim a variatione ipsius x nullam mutationem pati debeat, haec quantitas x quoque in eius determinationem plane non ingredietur.

COROLLARIUM 3

36. Hinc etiam patet aequationem differentialem $dz = adx + qdy$ realem esse non posse, nisi q sit functio ipsius y tantum. Quod etiam character supra expositus declarat; aequatione enim ad hanc formam $adx + qdy - dz = 0$ reducta ob $P = a$, $Q = q$ et $R = -1$ erit

$$L = \left(\frac{dq}{dz}\right), \quad M = 0 \quad \text{et} \quad N = -\left(\frac{dq}{dx}\right)$$

ideoque realitas postulat, ut sit $a\left(\frac{dq}{dz}\right) + \left(\frac{dq}{dx}\right) = 0$. At per hypothesin q non pendet a z, unde ob $\left(\frac{dq}{dz}\right) = 0$ erit $\left(\frac{dq}{dx}\right) = 0$ ideoque etiam q ab x non pendet.

SCHOLION 1

37. Ex allatis satis patet hanc operationem, qua functionem z determinavimus, veram esse integrationem, qua uti in vulgaribus integrationibus aliquid indeterminati introducitur. Hic scilicet ingressa est functio quaecunque ipsius y, cuius indoles per se nullo modo determinatur; eam quoque ita concipere licet, ut descripta curva quacunque, si eius abscissae per y indicentur, applicatae exhibeant eiusmodi functionem ipsius y. Neque vero opus est, ut haec curva sit regularis et aequatione quapiam contenta; sed curva quaecunque libero manus ductu descripta eundem praestat effectum, etiamsi sit maxime irregularis et ex pluribus partibus diversarum curvarum conflata. Huiusmodi functiones irregulares appellare licet discontinuas seu nexu continuitatis destitutas; unde hoc imprimis notatu dignum occurrit, quod, cum prioris generis integrationes alias functiones praeter continuas non admittant, hic etiam functiones discontinuae calculo subiiciantur, quod pluribus insignibus Geometris adeo calculi principiis adversari est visum.[1]) Verum integrati-

1) Vide notam paragrapho 299 adiectam. F. E.

onum in hoc secundo libro tradendarum vis praecipua in eo consistit, quod etiam functionum discontinuarum sint capaces; ex quo per hunc quasi novum calculum fines Analyseos maxime proferri sunt censendi.

SCHOLION 2

38. Quemadmodum deinde in vulgaribus integrationibus constans arbitraria ingressa semper ex indole problematis, cuius solutio eo perduxerat, determinatur, ita etiam hic natura problematis, cuius solutio huiusmodi integratione absolvitur, semper indolem functionis arbitrariae per integrationem ingressae determinabit. Ita si cordae tensae figura quaecunque inducatur eaque subito dimittatur, ut oscillationes peragat, ope principiorum mechanicorum ad quodvis tempus figura, quam corda tum sit habitura, definiri potest hocque fit eiusmodi integratione, qua functio quaedam arbitraria introducitur; quam autem deinceps ita determinari convenit, ut pro ipso motus initio ipsa illa figura cordae inducta prodeat; et cum solutio debeat esse generalis, ut satisfaciat figurae cuicunque initiali, necesse est, ut etiam ad eos casus pateat, quibus cordae initio figura irregularis nullo continuitatis nexu praedita inducatur, quod fieri non posset, nisi per integrationem eiusmodi functio arbitrio nostro relicta ingrederetur, quam etiam ad figuras irregulares adaptare liceret.[1]) Huiusmodi functiones arbitrarias, prouti hic feci, eiusmodi signandi modo $f:y$ indicabo, unde cavendum erit, ne littera f pro quantitate habeatur, quocirca ipsi *colon* suffigere visum est. Simili modo in sequentibus haec scriptio $f:(x+y)$ denotabit functionem arbitrariam quantitatis $x+y$; ac ubi plures tales functiones in calculum ingredientur, praeter litteram f etiam his characteribus φ, ψ, θ etc. cum simili significatione utar.

PROBLEMA 5

39. *Investigare indolem functionis z binarum variabilium x et y, ut formula differentialis $\left(\frac{dz}{dx}\right)=p$ aequalis fiat functioni datae ipsius x, quae sit X, ita ut sit $p=X$.*

SOLUTIO

Posito $dz=pdx+qdy$ ob $p=X$ erit $dz=Xdx+qdy$. Quia iam huius differentialis pars Xdx est data, ad integrale inveniendum accipiatur y con-

1) Vide notam paragrapho 299 adiectam. F. E.

stans, et cum sit $dz = Xdx$, erit integrando $z = \int Xdx +$ Const.; quae constans cum etiam quantitatem y utcunque implicare possit, pro ea assumere licebit functionem quamcunque arbitrariam ipsius y eritque ergo integrale quaesitum

$$z = \int Xdx + f:y,$$

quae per differentiationem praebet $dz = Xdx + dyf':y$, ita ut sit $q = f':y$ atque $\left(\frac{dz}{dx}\right) = X$, plane ut requirebatur.

COROLLARIUM 1

40. Aequationis ergo $\left(\frac{dz}{dx}\right) = X$ existente z functione duarum variabilium x et y integrale est $z = \int Xdx + f:y$, ubi ob X datum formula integralis $\int Xdx$ datam functionem ipsius x denotat, quandoquidem constans hac integratione ingressa in functione arbitraria $f:y$ comprehendi potest.

COROLLARIUM 2

41. Hinc sequitur aequationem differentialem $dz = Xdx + qdy$ realem esse non posse, nisi q sit functio ipsius y; quod quidem cum hac limitatione est intelligendum, nisi q etiam involvat quantitatem z; quem casum autem hinc removemus.

SCHOLION

42. Si enim q etiam a z pendere queat, aequatio $dz = Xdx + qdy$ realis erit, si q fuerit functio quaecunque binarum quantitatum $z - \int Xdx$ et y; id quod hinc facillime patet, si ponatur $z - \int Xdx = u$, ita ut iam q futura sit functio binarum quantitatum u et y. Tum enim aequatio differentialis, quae fit $du = qdy$, duas tantum continet variabiles u et y ideoque certo est realis; et quomodocunque eius integrale se habeat, inde semper u aequabitur certae functioni ipsius y, unde fit $u = z - \int Xdx = f:y$ prorsus ut ante. Quoties ergo esse debet $\left(\frac{dz}{dx}\right) = X$, etiam ne hoc quidem casu excepto, quo forte q ipsam quantitatem z implicat, integrale erit

$$z = \int Xdx + f:y$$

neque unquam alia solutio locum habere potest.

Erit ergo hoc integrale completum, propterea quod functionem arbitrariam involvit, id quod pro certissimo criterio integralis completi est habendum. Hic igitur ad integrale completum requiritur, ut non tam constans quaedam arbitraria, sed functio adeo variabilis arbitraria ingrediatur; ita, si quis pro casu $\left(\frac{dz}{dx}\right) = axx$ exhibeat hoc integrale

$$z = \frac{1}{3}ax^3 + A + By + Cy^2 + \text{etc.},$$

id tantum erit particulare, etiamsi plures constantes arbitrarias A, B, C etc. ac fortasse infinitas complectatur; verum enim integrale completum

$$z = \frac{1}{3}ax^3 + f:y$$

infinite latius patet; id quod ad sequentia recte intelligenda probe notari oportet.

Occurrent autem utique casus, quibus ob defectum methodi integrale completum investigandi integralibus particularibus contenti esse debemus, quae, etiamsi adeo infinitas constantes arbitrarias comprehendant, tamen pro solutionibus particularibus tantum sunt habenda. Hanc observationem in sequentibus perpetuo meminisse oportet, ne circa integralia particularia et completa unquam decipiamur.

PROBLEMA 6

43. *Si z debeat esse eiusmodi functio binarum variabilium x et y, ut formula differentialis $\left(\frac{dz}{dx}\right) = p$ aequetur functioni cuipiam datae ipsarum x et y, definire in genere indolem functionis quaesitae z.*

SOLUTIO

Sit V functio ista data ipsarum x et y, cui formula differentialis $\left(\frac{dz}{dx}\right) = p$ aequalis esse debet, ac posito $dz = pdx + qdy$ requiritur, ut sit $p = V$. Iam ad formam functionis z inveniendam consideretur quantitas y tanquam constans eritque $dz = Vdx$. Integretur igitur formula $\int Vdx$ spectata sola x ut variabili, quia y pro constante sumitur, ita ut in hac formula unica insit variabilis x ideoque eius integratio nulli obnoxia sit difficultati; id tantum est tenendum constantem integratione ingressam utcunque involvere posse

alteram quantitatem y sicque pro functione quaesita z haec habebitur expressio

$$z = \int V dx + f : y$$

integrali $\int V dx$ ita sumto, quasi quantitas y esset constans solaque x variabilis; at $f : y$ denotat functionem quamcunque arbitrariam ipsius y ne exclusis quidem formis discontinuis, quae nullis expressionibus analyticis exhiberi queant, atque ob hanc ipsam functionem arbitrariam integratio pro completa est habenda.

COROLLARIUM 1

44. Cum V sit functio data ipsarum x et y, formula integralis $\int V dx$ erit etiam functio cognita et determinata earundem quantitatum x et y; quod enim per integrationem arbitrarii ingreditur, in altera parte $f : y$ comprehenditur.

COROLLARIUM 2

45. Hinc etiam differentialis dz altera pars $q dy$ ex variabilitate ipsius y oriunda definitur. Nam per § 27 est formae $\int V dx$ differentiale ex utraque variabili x et y ortum

$$V dx + dy \int dx \left(\frac{dV}{dy}\right),$$

ac si functionis $f : y$ differentiale indicetur per $dy f' : y$, erit

$$dz = V dx + dy \int dx \left(\frac{dV}{dy}\right) + dy f' : y.$$

COROLLARIUM 3

46. Cum ergo posuerimus $dz = p dx + q dy$ sitque $p = V$, erit

$$q = \int dx \left(\frac{dV}{dy}\right) + f' : y,$$

ubi ob V functionem datam ipsarum x et y etiam $\left(\frac{dV}{dy}\right)$ erit functio data et in integratione $\int dx \left(\frac{dV}{dy}\right)$ sola x pro variabili habetur.

EXEMPLUM 1

47. *Quaeratur eiusmodi functio z ipsarum x et y, ut sit* $\left(\frac{dz}{dx}\right)=\frac{x}{\sqrt{(xx+yy)}}$.

Ob $V=\frac{x}{\sqrt{(xx+yy)}}$ erit

$$\int V dx=\sqrt{(xx+yy)}$$

ideoque habemus

$$z=\sqrt{(xx+yy)}+f:y,$$

unde fit

$$\left(\frac{dz}{dy}\right)=q=\frac{y}{\sqrt{(xx+yy)}}+f':y,$$

id quod etiam per regulam datam prodit. Erit enim

$$\left(\frac{dV}{dy}\right)=\frac{-xy}{(xx+yy)^{\frac{3}{2}}},$$

hinc sumta y constante

$$\int dx\left(\frac{dV}{dy}\right)=-y\int\frac{xdx}{(xx+yy)^{\frac{3}{2}}}=\frac{y}{\sqrt{(xx+yy)}}.$$

EXEMPLUM 2

48. *Quaeratur eiusmodi functio z ipsarum x et y, ut sit* $\left(\frac{dz}{dx}\right)=\frac{y}{\sqrt{(yy-xx)}}$.

Cum sit $V=\frac{y}{\sqrt{(yy-xx)}}$, erit

$$\int V dx=y\,\text{Ang. sin.}\,\frac{x}{y}$$

hincque

$$z=y\,\text{Ang. sin.}\,\frac{x}{y}+f:y;$$

cuius differentiale ex ipsius y variabilitate oriundum si desideremus, ob

$$\left(\frac{dV}{dy}\right)=\frac{-xx}{(yy-xx)^{\frac{3}{2}}}$$

erit

$$\int dx\left(\frac{dV}{dy}\right)=-\int\frac{xxdx}{(yy-xx)^{\frac{3}{2}}}=\int\frac{dx}{\sqrt{(yy-xx)}}-yy\int\frac{dx}{(yy-xx)^{\frac{3}{2}}}$$

ideoque

$$\int dx \left(\frac{dV}{dy}\right) = \text{Ang. sin.} \frac{x}{y} - \frac{x}{\sqrt{(yy-xx)}}$$

et

$$q = \text{Ang. sin.} \frac{x}{y} - \frac{x}{\sqrt{(yy-xx)}} + f' : y.$$

Idem reperitur ex differentiatione expressionis pro z inventae

$$dz = dy \,\text{Ang. sin.} \frac{x}{y} + \frac{ydx - xdy}{\sqrt{(yy-xx)}} + dyf' : y,$$

unde pro $q = \left(\frac{dz}{dy}\right)$ idem valor prodit.

EXEMPLUM 3

49. *Quaeratur eiusmodi functio z ipsarum x et y, ut sit* $\left(\frac{dz}{dx}\right) = \frac{a}{\sqrt{(aa-yy-xx)}}$.

Ob $V = \frac{a}{\sqrt{(aa-yy-xx)}}$ erit

$$\int V dx = a \,\text{Ang. sin.} \frac{x}{\sqrt{(aa-yy)}},$$

unde functionis z forma quaesita est

$$z = a \,\text{Ang. sin.} \frac{x}{\sqrt{(aa-yy)}} + f : y.$$

Deinde quia

$$\left(\frac{dV}{dy}\right) = \frac{ay}{(aa-yy-xx)^{\frac{3}{2}}},$$

erit

$$\int dx \left(\frac{dV}{dy}\right) = ay \int \frac{dx}{(aa-yy-xx)^{\frac{3}{2}}} = \frac{ay}{aa-yy} \cdot \frac{x}{\sqrt{(aa-yy-xx)}}$$

ideoque

$$\left(\frac{dz}{dy}\right) = q = \frac{axy}{(aa-yy)\sqrt{(aa-yy-xx)}} + f' : y,$$

quae eadem expressio etiam ex ipsa differentiatione ipsius z eruitur.

SCHOLION 1

50. In hoc calculo tamen adhuc quaedam incertitudo relinquitur, qua valor quantitatis q afficitur. Cum enim valor ipsius $z = \int V dx + f : y$ sit determinatus, quandoquidem integrale $\int V dx$ respectu ipsius x ita fuerit deter-

minatum, ut pro dato ipsius x valore etiam datum valorem obtineat, adeoque in eius differentiali pleno nulla incertitudo inesse potest, sed necesse est, ut valor ipsius q aeque prodeat determinatus atque ipsius p, interim tamen formula integralis $\int dx\left(\frac{dV}{dy}\right)$ non determinatur, sed novam arbitrariam a priori non pendentem introducere videtur. Ut igitur talis significatus vagus evitetur, spectari oportet conditionem, qua integrale $\int Vdx$ determinatur, eademque conditio in formulae $\int dx\left(\frac{dV}{dy}\right)$ integratione adhiberi debet. Nam ponamus integrale $\int Vdx$ ita capi, ut evanescat posito $x=a$, sitque eius valor determinatus $\int Vdx = S$, isque igitur potentia saltem habebit factorem $a-x$ seu $a^n - x^n$; qui cum non contineat y, etiam $\left(\frac{dS}{dy}\right)$ eundem factorem continebit ideoque $\left(\frac{dS}{dy}\right)$ evanescet posito $x=a$. Est vero $\left(\frac{dS}{dy}\right) = \int dx\left(\frac{dV}{dy}\right)$, ex quo perspicitur, si integrale $\int Vdx$ ita capiatur, ut evanescat posito $x=a$, etiam alterum integrale $\int dx\left(\frac{dV}{dy}\right)$ ita capi debere, ut evanescat posito $x=a$.

In allatis binis postremis exemplis utraque integratio ita est instituta, ut evanescat posito $x=0$, in primo autem nulla huiusmodi regula est observata; sin autem eandem legem adhibeamus, habebimus

$$\int Vdx = \sqrt{(xx+yy)} - y \quad \text{et} \quad \int dx\left(\frac{dV}{dy}\right) = \frac{y}{\sqrt{(xx+yy)}} - 1,$$

unde quidem eadem solutio emergit, quia ibi $-y$ continetur in $f:y$ et hic -1 in $f':y$. Perinde autem est, quacunque lege prior integratio determinetur, dummodo eadem lege et in posteriori utamur.

SCHOLION 2

51. Principium huius determinationis isto innititur theoremate aeque elegante ac notatu digno:

Si S sit eiusmodi functio binarum variabilium x et y, quae evanescat posito $x=a$, fueritque

$$dS = Pdx + Qdy,$$

tum etiam quantitas Q evanescet posito $x=a$.

Unde simul colligitur, si S evanescat posito $y=b$, tum etiam fieri $P=0$, si ponatur $y=b$. Hic autem probe observandum est, quae de simili deter-

minatione binarum formularum integralium $\int V dx$ et $\int dx \left(\frac{dV}{dy}\right)$ sunt praecepta, tantum valere, si valor a ipsi x tribuendus fuerit constans; neque etiam superius theorema locum habet, si verbi gratia functio S evanescat posito $x = y$; inde enim neutiquam sequitur eodem casu quantitatem Q esse evanituram. Etiamsi enim functio S factorem habeat $x - y$ vel $x^n - y^n$, minime sequitur formulam $\left(\frac{dS}{dy}\right)$ seu Q eundem factorem esse habituram, quemadmodum usu venit, si factor fuerit $x - a$ seu $x^n - a^n$.

Dixi autem non opus esse, ut talis factor revera adsit, dummodo quasi potentia in functione S contineatur. Veluti si fuerit

$$S = a - x + y - \sqrt{(aa - xx + yy)},$$

quae functio posito $x = a$ utique evanescit, etiamsi neque factorem $x - a$ neque $x^n - a^n$ contineat; simul vero etiam

$$\left(\frac{dS}{dy}\right) = 1 - \frac{y}{\sqrt{(aa - xx + yy)}}$$

posito $x = a$ evanescit.

In huiusmodi ergo calculo, quo in his problematibus utimur, ubi integrale formulae $\int V dx$ exhiberi debet, id semper ex duabus partibus compositum spectamus, altera indeterminata per functionem $f:y$ indicata, altera autem, quam proprie per $\int V dx$ exprimimus, determinata, quae scilicet posito $x = a$ evanescat; hicque semper perinde est, qualis constans pro a assumatur, dum discrimen perpetuo alteri parti indeterminatae involvitur.

PROBLEMA 7

52. *Si z debeat ita determinari per binas variabiles x et y, ut formula differentialis $\left(\frac{dz}{dx}\right) = p$ aequetur datae cuipiam functioni ipsarum y et z, quae sit $= V$, definire in genere indolem functionis z per x et y.*

SOLUTIO

Cum posito $dz = p dx + q dy$ sit $p = V$, si quantitatem y pro constante capiamus, erit $dz = V dx$; ubi cum V sit functio data ipsarum y et z et y

pro constante habeatur, aequatio $\frac{dz}{V} = dx$ erit integrabilis, ex cuius integratione completa oritur

$$\int \frac{dz}{V} = x + f : y,$$

qua aequatione relatio inter ternas variabiles x, y et z ita in genere exprimitur, ut ex ea z per x et y definiri indolesque functionis z assignari possit.

Quodsi hinc alteram quoque differentialis partem $q dy$ seu functionem $q = \left(\frac{dz}{dy}\right)$ indagare velimus, ponamus integrale $\int \frac{dz}{V}$, ubi y ut constans spectatur, ita capi, ut evanescat posito $z = c$, eritque quantitatem $\int \frac{dz}{V}$ denuo differentiando, ut etiam y variabilis assumatur,

$$d. \int \frac{dz}{V} = \frac{dz}{V} + dy \int dz \left(\frac{d(1:V)}{dy}\right)$$

seu

$$d. \int \frac{dz}{V} = \frac{dz}{V} - dy \int \frac{dz}{VV} \left(\frac{dV}{dy}\right),$$

ubi in integrali $\int \frac{dz}{VV} \left(\frac{dV}{dy}\right)$ quantitas y iterum pro constante habetur hocque integrale ita capi debet, ut posito $z = c$ evanescat. Quo facto cum aequationis inventae differentiale sit

$$\frac{dz}{V} - dy \int \frac{dz}{VV} \left(\frac{dV}{dy}\right) = dx + dy f' : y,$$

pro forma proposita habebimus

$$dz = V dx + dy \left(V \int \frac{dz}{VV} \left(\frac{dV}{dy}\right) + V f' : y\right),$$

unde quantitas q innotescit.

COROLLARIUM 1

53. In hoc problemate facillime definitur, qualis functio quantitas x futura sit binarum reliquarum y et z, cum sit

$$x = \int \frac{dz}{V} - f : y,$$

siquidem V per y et z detur. Perinde autem est, sive z per x et y sive x per y et z determinetur.

COROLLARIUM 2

54. Cum relatio inter ternas variabiles x, y et z ita sit determinata, ut fiat $\left(\frac{dz}{dx}\right) = V$ functioni datae ipsarum y et z, ob $dx = \frac{dz}{V}$ sumto y constante erit x eiusmodi functio ipsarum y et z, ut sit $\left(\frac{dx}{dz}\right) = \frac{1}{V}$ ideoque

$$\left(\frac{dz}{dx}\right) \cdot \left(\frac{dx}{dz}\right) = 1.$$

SCHOLION

55. In genere autem quaecunque relatio inter ternas variabiles x, y et z proponatur, unde unaquaeque per binas reliquas determinari et tanquam earundem functio spectari possit, semper erit $\left(\frac{dz}{dx}\right) \cdot \left(\frac{dx}{dz}\right) = 1$. Ponamus enim aequatione illam relationem exprimente differentiata prodire

$$Pdx + Qdy + Rdz = 0$$

ac manifestum est sumta y pro constante fore

$$Pdx + Rdz = 0$$

ideoque tam $\left(\frac{dz}{dx}\right) = \frac{-P}{R}$ quam $\left(\frac{dx}{dz}\right) = \frac{-R}{P}$; simili autem modo erit

$$\left(\frac{dx}{dy}\right) = \frac{-Q}{P}, \quad \left(\frac{dy}{dx}\right) = \frac{-P}{Q}, \quad \left(\frac{dz}{dy}\right) = \frac{-Q}{R}, \quad \left(\frac{dy}{dz}\right) = \frac{-R}{Q},$$

unde propositum patet, etiamsi relatio inter plures tribus variabiles locum habeat.

Ceterum hic casus a praecedentibus differt, quod hic natura functionis z, quatenus ex binis reliquis x et y formatur, non explicite exhibeatur, sed per resolutionem demum aequationis inventae definiri debet, cuius rei aliquot exempla evolvisse iuvabit.

EXEMPLUM 1

56. *Quaeratur eiusmodi functio z ipsarum x et y, ut sit* $\left(\frac{dz}{dx}\right) = \frac{y}{z}$.

Cum ergo sit $dz = \frac{ydx}{z} + qdy$, erit y pro constante sumendo $zdz = ydx$ et

$$\frac{1}{2}zz = xy + f:y.$$

Pro q inveniendo differentietur haec aequatio generaliter

$$zdz = ydx + xdy + dyf':y$$

eritque

$$q = \frac{x}{z} + \frac{1}{z}f':y,$$

quod idem per regulam datam reperitur. Nam ob $V = \frac{y}{z}$ erit $\int \frac{dz}{V} = \frac{zz}{2y}$ integrali ita sumto, ut evanescat posito $z = 0$, tum vero ob $\left(\frac{dV}{dy}\right) = \frac{1}{z}$ erit

$$\int \frac{dz}{VV}\left(\frac{dV}{dy}\right) = \int \frac{zdz}{yy} = \frac{zz}{2yy}$$

eadem integrationis lege observata. Hinc fit $dz = \frac{ydx}{z} + \frac{ydy}{z}\left(\frac{zz}{2yy} + f':y\right)$ et

$$q = \frac{z}{2y} + \frac{y}{z}f':y,$$

quae expressio cum praecedente convenit; ex comparatione enim fit

$$x + f':y = \frac{zz}{2y} + yf':y,$$

unde x aequatur ut ante quantitati $\frac{zz}{2y}$ una cum functione ipsius y. Hoc tantum notetur, quod ad consensum perfectum hic pro $f:y$ scribere debuissemus $yf:y$.

EXEMPLUM 2

57. *Quaeratur eiusmodi functio z binarum variabilium x et y, ut sit*

$$\left(\frac{dz}{dx}\right) = \frac{\sqrt{(yy - zz)}}{z}.$$

Cum ergo sit $dz = \frac{dx\sqrt{(yy - zz)}}{z} + qdy$, sumta y constante fit

$$dx = \frac{zdz}{\sqrt{(yy - zz)}}$$

et integrando

$$x = y - \sqrt{(yy - zz)} - f:y;$$

unde vicissim differentiando oritur

$$dx = dy - \frac{ydy - zdz}{\sqrt{(yy - zz)}} - dyf' : y$$

seu

$$dz = \frac{dx\sqrt{(yy - zz)}}{z} + dy\left(\frac{y}{z} - \frac{\sqrt{(yy - zz)}}{z}(1 - f' : y)\right).$$

Per regulam autem datam ob $V = \frac{\sqrt{(yy-zz)}}{z}$ est $\int \frac{dz}{V} = y - \sqrt{(yy - zz)}$ integrali ita sumto, ut evanescat posito $z = 0$. Iam vero est

$$\left(\frac{dV}{dy}\right) = \frac{y}{z\sqrt{(yy-zz)}} \quad \text{et} \quad \frac{1}{VV}\left(\frac{dV}{dy}\right) = \frac{yz}{(yy-zz)^{\frac{3}{2}}},$$

hinc

$$\int \frac{dz}{VV}\left(\frac{dV}{dy}\right) = \frac{y}{\sqrt{(yy-zz)}} - 1$$

integrali eadem lege sumto. Quocirca colligitur

$$q = \frac{\sqrt{(yy-zz)}}{z}\left(\frac{y}{\sqrt{(yy-zz)}} - 1 + f' : y\right) = \frac{y}{z} - \frac{\sqrt{(yy-zz)}}{z}(1 - f' : y)$$

prorsus ut ante.

PROBLEMA 8

58. *Si z ita debeat determinari per binas variabiles x et y, ut formula differentialis $\left(\frac{dz}{dx}\right) = p$ aequetur functioni cuipiam datae ipsarum x et z, quae sit $= V$, definire in genere indolem functionis z per x et y.*

SOLUTIO

Ponatur $dz = pdx + qdy$, et cum sit $p = V$, sumatur quantitas y constans eritque $dz - Vdx = 0$, quae aequatio duas tantum quantitates variabiles x et z continens integrabilis reddetur ope cuiusdam multiplicatoris, qui sit $= M$, ita ut $Mdz - MVdx$ sit differentiale verum cuiuspiam functionis ipsarum x et z, quae functio sit $= S$ quantitatem y non involvens. Ex quo aequatio nostra integralis erit $S = f : y$, unde indoles functionis z, quemadmodum per x et y determinatur, innotescit. Differentiemus hanc aequationem

sumto praeter x et z etiam y variabili eritque

$$dS = Mdz - MVdx = dyf' : y$$

seu

$$dz = Vdx + \frac{dy}{M} f' : y,$$

ita ut sit $q = \frac{1}{M} f' : y$.

COROLLARIUM 1

59. Multiplicator etiam M formulam $dz - Vdx$ integrabilem reddens quantitatem y non continebit, quia in functione data V non inest y. Statim autem hoc multiplicatore invento valor ipsius $q = \frac{1}{M} f' : y$ colligitur.

COROLLARIUM 2

60. Si formulae differentialis $Mdz - MVdx$ integrale fuerit S, functio ipsarum x et z, pro solutione problematis habebimus $S = f : y$, unde patet constantem, quam quis forte ad S adiicere voluerit, iam in functione arbitraria $f : y$ contineri.

EXEMPLUM 1

61. *Quaeratur eiusmodi functio z ipsarum x et y, ut sit* $\left(\frac{dz}{dx}\right) = \frac{nz}{x}$.

Posito $dz = \frac{nzdx}{x} + qdy$ sumto y constante erit $dz - \frac{nzdx}{x} = 0$, quae aequatio per $\frac{1}{z}$ multiplicata fit integrabilis, ita ut sit multiplicator $M = \frac{1}{z}$ hincque integrale $S = lz - lx^n$; ergo aequatio nostra integralis quaesita erit $l\frac{z}{x^n} = f : y$, unde etiam $\frac{z}{x^n}$ aequabitur functioni cuicunque ipsius y, ita ut sit $z = x^n f : y$.

EXEMPLUM 2

62. *Quaeratur eiusmodi functio z binarum variabilium x et y, ut sit formula differentialis* $\left(\frac{dz}{dx}\right) = nx - z$.

Posito $dz = (nx - z)dx + qdy$ sumto y constante erit $dz + zdx - nxdx = 0$, quae ope multiplicatoris $M = e^x$ dat

$$S = e^x z - n\int e^x xdx = e^x z - ne^x x + ne^x,$$

unde aequatio quaesitam relationem inter x, y et z exprimens est

$$e^x z - ne^x x + ne^x = f:y \quad \text{sive} \quad z = n(x-1) + e^{-x} f:y,$$

tum vero erit

$$q = \left(\frac{dz}{dy}\right) = e^{-x} f':y.$$

EXEMPLUM 3

63. *Quaeratur eiusmodi functio z binarum variabilium x et y, ut sit formula differentialis* $\left(\frac{dz}{dx}\right) = \frac{xz}{xx+zz}$.

Ponatur ergo $dz = \frac{xz\,dx}{xx+zz} + q\,dy$ et posito y constante quaeratur integrale huius aequationis differentialis

$$dz - \frac{xz\,dx}{xx+zz} = 0,$$

quae integrabilis redditur ope cuiusdam divisoris, qui ob homogeneitatem reperitur scribendo x et z loco differentialium dx et dz, ita ut hic divisor sit

$$z - \frac{xxz}{xx+zz} = \frac{z^3}{xx+zz}$$

hincque multiplicator $M = \frac{xx+zz}{z^3}$. Quare erit

$$dS = \frac{(xx+zz)dz}{z^3} - \frac{x\,dx}{zz}$$

ideoque

$$S = \frac{-xx}{2zz} + lz,$$

unde aequatio nostra quaesita erit

$$lz - \frac{xx}{2zz} = f:y \quad \text{et} \quad q = \frac{z^3}{xx+zz} f':y,$$

ex qua, cum posito $lz - \frac{xx}{2zz} = u$ sit $u = f:y$, etiam vicissim concludi potest fore $y = f:u$.

PROBLEMA 9

64. *Si z ita debeat determinari per binas variabiles x et y, ut formula differentialis $\left(\frac{dz}{dx}\right)$ aequetur functioni cuipiam datae omnes tres variabiles x, y et z implicanti, quae sit $= V$, definire in genere indolem functionis z per x et y.*

SOLUTIO

Cum sit $dz = Vdx + qdy$, si sumamus y constans, erit $dz = Vdx$, quae ergo aequatio duas tantum continet variabiles x et z, litteram autem y in functione V involvens. Dabitur ergo multiplicator M hanc aequationem integrabilem reddens, ita ut sit

$$Mdz - MVdx = dS,$$

unde aequatio integralis relationem inter x, y et z exprimens erit $S = f:y$, ubi S erit functio certa ipsarum x, y et z, fierique potest, ut etiam M omnes has tres litteras comprehendat. Convenit autem functioni S per integrationem inventae valorem determinatum tribui, quoniam pars indeterminata in functione arbitraria $f:y$ includitur. Ponamus ergo S ita capi, ut evanescat, si ponatur $x = a$ et $z = c$.

Quodsi hinc aequationis differentialis propositae alteram partem qdy invenire velimus, differentiemus functionem S sumto etiam y variabili sitque

$$dS = Mdz - MVdx + Qdy = dyf':y;$$

ubi cum sit

$$\left(\frac{dQ}{dz}\right) = \left(\frac{dM}{dy}\right) \quad \text{et} \quad \left(\frac{dQ}{dx}\right) = -\left(\frac{d.MV}{dy}\right),$$

erit sumto iterum y constante

$$dQ = dz\left(\frac{dQ}{dz}\right) + dx\left(\frac{dQ}{dx}\right) = dz\left(\frac{dM}{dy}\right) - dx\left(\frac{d.MV}{dy}\right),$$

quae formula certo erit integrabilis. Capi autem Q eadem lege debet, qua S sumsimus, ita ut evanescat posito $x = a$ et $z = c$, atque inventa hac quantitate Q, cum habeamus

$$dz = Vdx - \frac{Qdy}{M} + \frac{dy}{M}f':y,$$

erit

$$q = \left(\frac{dz}{dy}\right) = \frac{-Q + f':y}{M}.$$

Haec determinatio isto nititur fundamento, quod, si S fuerit eiusmodi functio ipsarum x, y et z, quae posito $x = a$ et $z = c$ evanescat, etiam formula differentialis $\left(\frac{dS}{dy}\right)$ eodem casu evanescat.

COROLLARIUM 1

65. Reducitur ergo resolutio huius problematis ad integrationem aequationis differentialis

$$dz - Vdx = 0,$$

in qua quantitas y ut constans spectatur, etiamsi V contineat omnes tres litteras x, y et z. Dabitur ergo utique multiplicator M, qui hanc aequationem integrabilem reddat, ut sit

$$Mdz - MVdx = dS$$

existente S certa quadam functione ipsarum x, y et z.

COROLLARIUM 2

66. Invento autem hoc multiplicatore M indeque integrali S quantitas z ita per binas variabiles x et y definietur, ut sit $S = f:y$, ubi $f:y$ denotat functionem quamcunque ipsius y sive continuam sive etiam discontinuam, ob cuius naturam integratio pro completa est habenda.

COROLLARIUM 3

67. Cum hoc modo relatio inter z, x, y fuerit definita, erit ea ita differentiata, ut omnes tres litterae x, y et z variabiles sumantur,

$$dz = Vdx + \frac{f':y - Q}{M}\,dy,$$

ubi quantitas Q ex suo differentiali

$$dQ = dz\left(\frac{dM}{dy}\right) - dx\left(\frac{d.MV}{dy}\right)$$

definiri debet sumta y constante, integrationem ita temperando, ut, si S evanescat casu $x = a$ et $z = c$, etiam Q eodem casu evanescat.

SCHOLION

68. Hic ergo ad insigne hoc theorema deducimur:

Quodsi fuerit S eiusmodi functio ipsarum x, y et z, quae evanescat ponendo $x = a$ et $z = c$, tum etiam pro eadem positione formulam $\left(\frac{dS}{dy}\right)$ esse evanituram.

Veluti si fuerit

$$S = Axx + Bxyz + Czz - Aaa - Bacy - Ccc,$$

erit

$$\left(\frac{dS}{dy}\right) = Bxz - Bac,$$

quarum utraque expressio casu $x = a$ et $z = c$ evanescit. Pluribus autem huiusmodi exemplis evolutis veritas theorematis ita patet, ut demonstratio solennis non desideretur. Interim huiusmodi functio semper quantitates solam y continentes a reliquis separando ita evolvi potest, ut in talem formam transmutetur

$$S = PY + QY' + RY'' + \text{etc.},$$

ubi per hypothesin P, Q, R etc. sunt functiones ipsarum x et z tantum et tales quidem, quae ponendo $x = a$ et $z = c$ singulae evanescant. Hinc iam perspicuum est fore

$$\left(\frac{dS}{dy}\right) = P\frac{dY}{dy} + Q\frac{dY'}{dy} + R\frac{dY''}{dy} + \text{etc.},$$

quae forma manifesto sub iisdem conditionibus evanescit. Quomodocunque autem functio S hac indole praedita fuerit complicata tam formulis irrationalibus quam transcendentibus, eam semper in eiusmodi formam evolvere licet; quae etsi in infinitum progrediatur, haec demonstratio tamen vim suam retinet.

EXEMPLUM 1

69. *Quaeratur eiusmodi functio z duarum variabilium x et y, ut sit formula differentialis* $\left(\frac{dz}{dx}\right) = \frac{xz}{ay}$.

Ponamus ergo $dz = \frac{xzdx}{ay} + qdy$ et sumta y constante habebitur aequatio $dz - \frac{xzdx}{ay} = 0$, ut sit $V = \frac{xz}{ay}$, et multiplicator erit $M = \frac{1}{z}$; unde fit

$$S = l\frac{z}{c} - \frac{xx - aa}{2ay}$$

et aequatio integralis completa functionem z determinans erit

$$l\frac{z}{c} + \frac{aa - xx}{2ay} = f:y.$$

Porro ad quantitatem q inveniendam ob $M = \frac{1}{z}$ et $MV = \frac{x}{ay}$ erit[1]) $dQ = \frac{xdx}{ayy}$ et $Q = \frac{xx - aa}{2ayy}$, unde fit $q = zf':y - \frac{z(xx - aa)}{2ayy}$.

Hic idem autem valor ex differentiatione aequationis inventae eruitur, quae praebet

$$\frac{dz}{z} - \frac{xdx}{ay} - \frac{aa - xx}{2ayy}dy = dyf':y$$

ideoque

$$dz = \frac{xzdx}{ay} + \frac{z(aa - xx)}{2ayy}dy + zdyf':y,$$

ita ut sit

$$q = \frac{z(aa - xx)}{2ayy} + zf':y.$$

EXEMPLUM 2

70. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{dz}{dx}\right) = \frac{y}{x + z}.$$

Cum sit $V = \frac{y}{x + z}$, habebitur sumto y constante haec aequatio

$$dz - \frac{ydx}{x + z} = 0,$$

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1) Editio princeps: $dQ = 0$ *et* $Q = 0$; *unde fit* $q = zf':y$. Quamobrem etiam formulae sequentes huius paragraphi corrigendae erant. F. E.

ad cuius multiplicatorem inveniendum multiplicetur primo per $x+z$, ut prodeat

$$xdz + zdz - ydx = 0 \quad \text{seu} \quad dx - \frac{xdz}{y} = \frac{zdz}{y},$$

quae multiplicata per $e^{-\frac{z}{y}}$ integrabilis evadit proditque

$$e^{-\frac{z}{y}}x = \int e^{-\frac{z}{y}}\frac{zdz}{y} = -e^{-\frac{z}{y}}z + \int e^{-\frac{z}{y}}dz$$

hincque

$$e^{-\frac{z}{y}}x = -e^{-\frac{z}{y}}z - ye^{-\frac{z}{y}} + C.$$

Quocirca erit multiplicator

$$M = (x+z)\cdot -\frac{1}{y}\cdot e^{-\frac{z}{y}} = -\frac{x+z}{y}e^{-\frac{z}{y}}$$

et

$$S = e^{-\frac{z}{y}}(x+z+y) - e^{-\frac{c}{y}}(a+c+y),$$

ex quo aequatio integralis completa erit

$$e^{-\frac{z}{y}}(x+z+y) - e^{-\frac{c}{y}}(a+c+y) = f:y.$$

Nunc porro cum sit $MV = -e^{-\frac{z}{y}}$, erit

$$\left(\frac{dM}{dy}\right) = e^{-\frac{z}{y}}\left(\frac{x+z}{yy} - \frac{z(x+z)}{y^3}\right) = e^{-\frac{z}{y}}(x+z)\left(\frac{1}{yy} - \frac{z}{y^3}\right)$$

et

$$\left(\frac{d.MV}{dy}\right) = -e^{-\frac{z}{y}}\cdot\frac{z}{yy}$$

hincque

$$dQ = e^{-\frac{z}{y}}\left(dz(x+z)\left(\frac{1}{yy} - \frac{z}{y^3}\right) + \frac{zdx}{yy}\right)$$

sumto y constante, unde integrando obtinebitur

$$Q = e^{-\frac{z}{y}}\left(\frac{xz}{yy} + 1 + \frac{z}{y} + \frac{zz}{yy}\right) - e^{-\frac{c}{y}}\left(\frac{ac}{yy} + 1 + \frac{c}{y} + \frac{cc}{yy}\right),$$

hinc

$$q = \frac{z}{y} + \frac{y+z}{x+z} - e^{\frac{z-c}{y}}\left(\frac{ac+cc+cy+yy}{y(x+z)}\right) - \frac{y}{x+z}e^{\frac{z}{y}}f':y,$$

ita ut sit

$$dz = \frac{y\,dx}{x+z} + q\,dy.$$

Aequatio autem inventa si differentietur, dat

$$-e^{-\frac{z}{y}}\frac{(x+z)dz}{y} + e^{-\frac{z}{y}}dx + e^{-\frac{z}{y}}dy\left(1+\frac{z}{y}+\frac{xz}{yy}+\frac{zz}{yy}\right)$$

$$-e^{-\frac{c}{y}}dy\left(1+\frac{c}{y}+c\frac{(a+c)}{yy}\right) = dy f':y,$$

unde idem prorsus valor pro q concluditur.

EXEMPLUM 3

71. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{dz}{dx}\right) = \frac{yy+zz}{yy+xx}.$$

Posito $dz = \frac{yy+zz}{yy+xx}dx + q\,dy$ sumatur quantitas y constans, et cum sit $dz - \frac{(yy+zz)dx}{yy+xx} = 0$, evidens est multiplicatorem idoneum esse $M = \frac{y}{yy+zz}$, unde, cum sit $\frac{y\,dz}{yy+zz} - \frac{y\,dx}{yy+xx} = 0$, erit per integrationem

$$S = \text{A tang.}\frac{z}{y} - \text{A tang.}\frac{x}{y} + C = \text{A tang.}\frac{yz-yx}{yy+xz} - \text{A tang.}\frac{(c-a)y}{ac+yy}$$

et functio quaesita z hac aequatione definietur

$$\text{A tang.}\frac{y(z-x)}{yy+xz} - \text{A tang.}\frac{(c-a)y}{ac+yy} = f:y.$$

Cum porro sit $MV = \frac{y}{yy+xx}$, erit

$$\left(\frac{dM}{dy}\right) = \frac{zz-yy}{(yy+zz)^2} \quad \text{et} \quad \left(\frac{d.MV}{dy}\right) = \frac{xx-yy}{(yy+xx)^2}$$

hincque

$$dQ = \frac{(zz-yy)dz}{(yy+zz)^2} - \frac{(xx-yy)dx}{(yy+xx)^2}$$

sumto y constante. Ergo

$$Q = \frac{-z}{yy+zz} + \frac{x}{yy+xx} + \frac{c}{yy+cc} - \frac{a}{yy+aa} \quad \text{et} \quad q = \frac{-Q+f':y}{M},$$

qui idem valor etiam ex differentiatione prodit.

Ceterum cum constantes a et c pro lubitu accipi queant, sumtis iis nihilo aequalibus seu saltem $c = a$ erit aequatio integralis

$$\text{A tang.}\,\frac{y(z-x)}{yy+xz} = f:y,$$

unde erit etiam

$$\frac{y(z-x)}{yy+xz} = \text{funct.}\,y \quad \text{et} \quad \frac{yy+xz}{z-x} = \text{funct.}\,y;$$

quae functio si dicatur Y, habebitur

$$z = \frac{yy+xY}{Y-x}.$$

SCHOLION

72. Vix opus est notari saepe fieri posse, ut solutio huiusmodi quaestionum superet vires Analyseos, quando scilicet aequatio differentialis resolvenda artificiis adhuc cognitis integrari nequit. Veluti si proponatur casus

$$\left(\frac{dz}{dx}\right) = \frac{yy}{xx+zz},$$

unde sumto y constante fieri debet $yydx = xxdz + zzdz$, cuius integrationem nondum expedire licet. Interim quia integrale per seriem exhiberi potest, modo id fiat complete, etiam solutio per seriem obtinebitur. Posito scilicet $x = \frac{-yydu}{udz}$ et sumto elemento dz constante oritur haec aequatio differentio-differentialis

$$y^4ddu + uzzdz^2 = 0,$$

unde per series integrando reperitur[1])

$$u = A\left(1 - \frac{z^4}{3\cdot 4y^4} + \frac{z^8}{3\cdot 4\cdot 7\cdot 8y^8} - \text{etc.}\right) + Bz\left(1 - \frac{z^4}{4\cdot 5y^4} + \frac{z^8}{4\cdot 5\cdot 8\cdot 9y^8} - \text{etc.}\right),$$

ubi pro A et B functiones quaecunque ipsius y accipi possunt. Quare si ponatur $\frac{A}{B} = f:y$, erit

$$x = \frac{yyf:y\left(\frac{z^3}{3y^4} - \frac{z^7}{3\cdot 4\cdot 7y^8} + \text{etc.}\right) - yy\left(1 - \frac{z^4}{4y^4} + \frac{z^8}{4\cdot 5\cdot 8y^8} - \text{etc.}\right)}{f:y\left(1 - \frac{z^4}{3\cdot 4y^4} + \frac{z^8}{3\cdot 4\cdot 7\cdot 8y^8} - \text{etc.}\right) + z\left(1 - \frac{z^4}{4\cdot 5y^4} + \frac{z^8}{4\cdot 5\cdot 8\cdot 9y^8} - \text{etc.}\right)},$$

1) Vide *Institutionum calculi integralis* vol. II, § 929, *Leonhardi Euleri Opera omnia*, series I, vol. 12, p. 147. F. E.

qua aequatione functio quaesita z per binas variabiles x et y generalissime exprimitur.

Quoniam ergo methodos aperuimus aequationes differentiales quascunque per approximationes integrandi idque complete[1]), his methodis in subsidium vocandis omnia problemata huc pertinentia saltem per approximationem resolvi poterunt. Ceterum in hac parte Analyseos sublimiori resolutionem aequationum differentialium ad priorem partem Analysis pertinentium pro concessa assumere possumus, omnino uti, quo longius in Analysi progredimur, ea semper, quae ad partes praecedentes pertinent, etiamsi non penitus sunt evoluta, tanquam confecta spectare solemus.

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1) Vide *Institutionum calculi integralis* vol. I, § 650—667, LEONHARDI EULERI *Opera omnia*, series I, vol. 11, p. 424—434. F. E.

CAPUT III

DE RESOLUTIONE AEQUATIONUM QUIBUS BINARUM FORMULARUM DIFFERENTIALIUM ALTERA PER ALTERAM UTCUNQUE DATUR

PROBLEMA 10

73. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut formulae differentiales $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ inter se fiant aequales, indolem istius functionis in genere determinare.*

SOLUTIO

Ponatur $\left(\frac{dz}{dx}\right)=p$ et $\left(\frac{dz}{dy}\right)=q$, ut sit $dz=pdx+qdy$ haecque formula $pdx+qdy$ integrationem sponte admittat. Quoniam igitur requiritur, ut sit $q=p$, erit $dz=p(dx+dy)$ et posito $x+y=u$ fiet $dz=pdu$; quae formula cum debeat esse per se integrabilis, necesse est, ut p sit functio quantitatis variabilis u nullam praeterea aliam variabilem involvens; hincque integrando ipsa quantitas $z=\int pdu$ aequabitur functioni ipsius u seu prodibit $z=f:u$, quae functio omnino arbitrio nostro relinquitur, ita ut pro z functio quaecunque ipsius u sive continua sive etiam discontinua assumta problemati satisfaciat. Quare cum sit $u=x+y$, erit pro solutione nostri problematis $z=f:(x+y)$. Quae forma, quo facilius appareat, quomodo conditioni praescriptae satisfaciat, sit $d.f:u=duf':u$ ideoque ob $u=x+y$ erit

$$dz=(dx+dy)f':(x+y)=dxf':(x+y)+dyf':(x+y)$$

ideoque et

$$\left(\frac{dz}{dx}\right)=p=f':(x+y) \quad \text{et} \quad \left(\frac{dz}{dy}\right)=q=f':(x+y)$$

ac propterea $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$ seu $q=p$, omnino uti problema postulat.

COROLLARIUM 1

74. Quaecunque ergo functio quantitatis $x+y$ formetur, ea pro z assumta praescriptam habebit proprietatem, ut sit $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$. Talem autem functionem indicamus signo $f:(x+y)$, ita ut sit $z=f:(x+y)$.

COROLLARIUM 2

75. Geometrice haec solutio ita referri potest. Descripta super axe linea curva quacunque sive regulari sive irregulari si abscissa exprimatur per $x+y$, applicata semper idoneum valorem pro functione z exhibebit.

COROLLARIUM 3

76. Universalitas huius solutionis per integrationem erutae in hoc consistit, quod quantitatis $x+y$ functionem qualemcunque sive continuam sive etiam discontinuam pro z invenerimus, quippe quae conditioni problematis semper satisfacit.

SCHOLION 1

77. Fundamentum solutionis hoc nititur principio, quod formula differentialis pdu integrabilis esse nequeat, nisi quantitas p sit functio ipsius u vel vicissim u functio ipsius p, ita ut nulla alia variabilis in computum ingrediatur. Quin etiam, qualiscunque fuerit p functio ipsius u, integrale, nisi actu exhiberi, semper tamen concipi potest. Si enim u denotet abscissam et p applicatam curvae cuiuscunque sive regularis sive irregularis, qua ratione utique functio quaecunque ipsius u in sensu latissimo repraesentari potest, eius curvae area $\int pdu$ praebet valorem formulae integralis $\int pdu$, quae iterum ut functio ipsius u spectari potest; ex quo vicissim functio quaecunque ipsius u naturam formulae integralis $\int pdu$ exhaurit. Quod autem functio quaecunque quantitatis $x+y$ pro z assumta satisfaciat conditioni, ut in differentiali $dz=pdx+qdy$ fiat $p=q$ seu $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$, ita per se est perspicuum, ut illustratione per exempla non egeat. Si enim verbi gratia ponatur

$$z=aa+b(x+y)+(x+y)^2=aa+bx+by+xx+2xy+yy,$$

erit differentiando

$$\left(\frac{dz}{dx}\right)=b+2x+2y \quad \text{et} \quad \left(\frac{dz}{dy}\right)=b+2x+2y,$$

qui valores inter se utique sunt aequales.

SCHOLION 2

78. Cum z sit functio binarum variabilium x et y ac ponatur

$$dz = pdx + qdy,$$

ut sit $\left(\frac{dz}{dx}\right) = p$ et $\left(\frac{dz}{dy}\right) = q$, in hoc capite eiusmodi quaestiones evolvere est propositum, in quibus aequatio quaecunque inter p et q praescribitur, in quam reliquarum variabilium x, y et z nulla ingrediatur. Proposita ergo aequatione quacunque inter binas formulas p et q et constantes quaeri oportet indolem functionis z binarum variabilium x et y, ut formulis inde per differentiationem natis $p = \left(\frac{dz}{dx}\right)$ et $q = \left(\frac{dz}{dy}\right)$ praescripta illa conditio conveniat. Quam tractationem quidem exorsi sumus ab exemplo simplicissimo $p = q$, cuius solutio etiam ope principii modo expositi confici potest. At vero idem principium sufficit problemati sequenti latius patenti resolvendo.

PROBLEMA 11

79. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut fiat $\alpha\left(\frac{dz}{dx}\right) + \beta\left(\frac{dz}{dy}\right) = \gamma$, indolem istius functionis z in genere definire.*

SOLUTIO

Posito $dz = pdx + qdy$ requiritur, ut sit $\alpha p + \beta q = \gamma$. Hinc cum sit $q = \frac{\gamma - \alpha p}{\beta}$, erit

$$dz = pdx + \frac{(\gamma - \alpha p)}{\beta}dy \quad \text{seu} \quad dz = \frac{\gamma}{\beta}dy + \frac{p}{\beta}(\beta dx - \alpha dy),$$

quam formulam integrabilem esse oportet. Cum autem pars $\frac{\gamma}{\beta}dy$ per se sit integrabilis, altera pars etiam integrabilis sit necesse est, unde posito $\beta x - \alpha y = u$, ut altera pars fiat $\frac{p}{\beta}du$, evidens est p functionem esse debere ipsius u indeque etiam integrale proditurum esse functionem ipsius $u = \beta x - \alpha y$. Quare ponamus

$$\int p(\beta dx - \alpha dy) = f:(\beta x - \alpha y)$$

eritque

$$z = \frac{\gamma}{\beta}y + \frac{1}{\beta}f:(\beta x - \alpha y)$$

seu aequatio quaesita indolem functionis z determinans erit

$$\beta z = \gamma y + f : (\beta x - \alpha y)$$

denotante signo f: functionem quamcunque sive continuam sive discontinuam formulae suffixae $\beta x - \alpha y$. Atque indicando formulae $f : u$ differentiale per $du f' : u$ erit

$$p = f' : (\beta x - \alpha y) \quad \text{et} \quad q = \frac{\gamma}{\beta} - \frac{\alpha}{\beta} f' : (\beta x - \alpha y),$$

unde manifesto resultat $\alpha p + \beta q = \gamma$.

COROLLARIUM 1

80. Eodem solutio redit, si pro p eius valorem $p = \frac{\gamma - \beta q}{\alpha}$ substituamus, unde fit

$$dz = \frac{\gamma}{\alpha} dx + \frac{q}{\alpha}(\alpha dy - \beta dx)$$

hincque eodem modo

$$z = \frac{\gamma x}{\alpha} + \frac{1}{\alpha} f : (\alpha y - \beta x).$$

Etsi enim haec forma a praecedente differre videtur, tamen facile eo reducitur ponendo ibi

$$f : (\beta x - \alpha y) = \frac{\gamma(\beta x - \alpha y)}{\alpha} + \frac{\beta}{\alpha} \varphi : (\alpha y - \beta x),$$

quae forma utique est functio ipsius $\beta x - \alpha y$.

COROLLARIUM 2

81. Si ergo in forma $dz = p dx + q dy$ debeat esse $p + q = 1$, ob $\alpha = 1$, $\beta = 1$ et $\gamma = 1$ solutio huc redit, ut fiat

$$z = y + f : (x - y).$$

Constructa ergo curva quacunque si abscissae $x - y$ respondeat applicata v, erit $z = y + v$.

SCHOLION

82. Si alia proponatur relatio inter p et q, eadem methodo solutionem obtinere non licet, sed alio principio uti convenit, cuius quidem veritas ex

primis calculi integralis elementis est manifesta. Notari scilicet oportet esse

$$\int p\,dx = px - \int x\,dp$$

similique modo

$$\int q\,dy = qy - \int y\,dq,$$

ita ut, cum sit

$$z = \int (p\,dx + q\,dy),$$

futurum sit

$$z = px + qy - \int (x\,dp + y\,dq).$$

Quomodo autem hoc principium ad solutionem huiusmodi quaestionum, quae ad hoc caput sint referendae, applicandum sit, in sequentibus problematibus docebitur.

PROBLEMA 12

83. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito $dz = p\,dx + q\,dy$ fiat $pq = 1$, indolem istius functionis z in genere definire.*

SOLUTIO

Ex principio ante stabilito notemus fore

$$z = px + qy - \int (x\,dp + y\,dq).$$

Cum iam ob $pq = 1$ sit $q = \frac{1}{p}$, erit

$$z = px + \frac{y}{p} - \int \left(x\,dp - \frac{y\,dp}{pp}\right).$$

Integrabilis ergo esse debet haec forma $\int \left(x - \frac{y}{pp}\right) dp$. At in genere formula $\int u\,dp$ integrationem non admittit, nisi sit u functio ipsius p; quare in nostro casu necesse est sit quantitas $x - \frac{y}{pp}$ functio ipsius p tantum, unde etiam integrale $\int dp\left(x - \frac{y}{pp}\right)$ erit functio ipsius p tantum; quae si indicetur per $f:p$ eiusque differentiale per $dp f':p$, erit

$$z = px + \frac{y}{p} - f:p \quad \text{et} \quad x - \frac{y}{pp} = f':p.$$

Quare ad problema nostrum solvendum nova variabilis p introduci debet, ex qua cum altera y coniuncta binae reliquae x et z determinentur. Sumta scilicet variabili p eiusque functione quacunque $f:p$ indeque per differentiationem derivata $f':p$ capiatur primo

$$x=\frac{y}{pp}+f':p$$

indeque erit

$$z=\frac{2y}{p}+pf':p-f:p,$$

quae est solutio problematis quaesita generalis.

COROLLARIUM 1

84. Hic igitur functio quaesita z per ipsas variabiles x et y explicite evolvi nequit, propterea quod quantitatem p ex aequatione $x-\frac{y}{pp}=f':p$ in genere per x et y definire non licet.

COROLLARIUM 2

85. Nihilo vero minus solutio pro idonea et completa est habenda, quoniam introducendo novam variabilem p ex binis y et p a se invicem non pendentibus ambae reliquae x et z definiuntur.

COROLLARIUM 3

86. Si sumamus $f':p=\alpha+\frac{\beta}{pp}$, erit

$$f:p=\alpha p-\frac{\beta}{p} \quad \text{et} \quad x-\alpha=\frac{\beta+y}{pp},$$

hinc $p=\sqrt{\frac{\beta+y}{x-\alpha}}$; unde functio quaesita z ita se habebit

$$z=\frac{2y\sqrt{(x-\alpha)}}{\sqrt{(\beta+y)}}+\frac{\alpha y+\beta x}{\sqrt{(x-\alpha)(\beta+y)}}-\frac{\alpha y-\beta x+2\alpha\beta}{\sqrt{(x-\alpha)(\beta+y)}}$$

seu $z=2\sqrt{(x-\alpha)(y+\beta)}$, quae est solutio particularis, et simplicissima est $z=2\sqrt{xy}$.

SCHOLION 1

87. Quemadmodum solutio huius problematis ex alio principio est deducta, ita etiam forma solutionis a praecedentibus discrepat, quod hic aequationem inter x, y et z explicitam exhibere non liceat, sed nova variabilis p introducatur. Cum igitur ante una aequatio inter ternas variabiles x, y et z solutionem continuisset, nunc accedente nova variabili p solutio geminam aequationem inter has quatuor variabiles postulat sicque pro nostro casu invenimus

$$z = px + \frac{y}{p} - f:p \quad \text{et} \quad x - \frac{y}{pp} = f':p$$

existente $d.f:p = dpf':p$, ubi functionis signum indefinitum $f:$, quod etiam functiones discontinuas admittit, universalitatem solutionis praestat. Quodsi hinc litteram p eliminare liceret, aequatio evoluta inter x, y et z obtineretur; haec autem eliminatio succedit, quoties pro $f:p$ functio algebraica ipsius p assumitur, in genere autem nullo modo sperari potest. Nihilo vero minus ope curvae pro lubitu assumtae problema construi potest; sumta enim curva quacunque sive regulari sive irregulari ponatur abscissa $= p$ sitque applicata $f':p = r$; erit $f:p = \int rdp$ area eius curvae; quae si dicatur $= s$, aequationes binae

$$x - \frac{y}{pp} = r \quad \text{et} \quad z = px + \frac{y}{p} - s$$

solutionem completam problematis praebebunt. Scilicet sumto pro x valore quocunque erit $y = pp(x - r)$ hincque fit $z = 2px - pr - s$, in qua solutione nihil ad praxin spectans desiderari potest.

Hinc patet etiam fortasse fieri posse, ut duae novae variabiles sint introducendae ac tum solutio tribus aequationibus contineatur; neque etiam tum quicquam deerit ad usum practicum.

SCHOLION 2

88. Cum pro formula $dz = pdx + qdy$ requiratur, ut sit $pq = 1$, introducendo angulum indefinitum φ alia solutio concinnior elici potest. Posito enim $p = \text{tang.}\,\varphi$ erit $q = \cot.\,\varphi$ et ob $dz = dx\,\text{tang.}\,\varphi + dy\cot.\,\varphi$ fiet per reductionem supra indicatam

$$z = x\,\text{tang.}\,\varphi + y\cot.\,\varphi - \int d\varphi\left(\frac{x}{\cos.\varphi^2} - \frac{y}{\sin.\varphi^2}\right),$$

unde patet formulam $\frac{x}{\cos.\varphi^2}-\frac{y}{\sin.\varphi^2}$ esse debere functionem ipsius φ; quae si ponatur $f':\varphi$ et formula integralis $\int d\varphi f':\varphi = f:\varphi$, binae aequationes solutionem continentes erunt

$$\frac{x}{\cos.\varphi^2}-\frac{y}{\sin.\varphi^2}=f':\varphi \quad \text{et} \quad z = x\,\text{tang.}\,\varphi + y\cot.\varphi - f:\varphi,$$

unde iam pro lubitu x vel y eliminare licet. Quin etiam utramque eliminare possumus ac per binas variabiles z et φ binae reliquae x et y ita exprimentur

$$x = \tfrac{1}{2}z\cot.\varphi + \tfrac{1}{2}\cot.\varphi\cdot f:\varphi + \tfrac{1}{2}\cos.\varphi^2\cdot f':\varphi,$$

$$y = \tfrac{1}{2}z\,\text{tang.}\,\varphi + \tfrac{1}{2}\text{tang.}\,\varphi\cdot f:\varphi - \tfrac{1}{2}\sin.\varphi^2\cdot f':\varphi.$$

Quodsi igitur hinc differentialia capiantur ac ponatur $dy=0$, ex posteriori dabitur relatio inter dz et $d\varphi$, unde, si ipsius $d\varphi$ valor in priori substituatur, necesse est prodeat $dz = dx\,\text{tang.}\,\varphi$; simili autem modo si ponatur $dx=0$, ex altera orietur $dz = dy\cot.\varphi$.

PROBLEMA 13

89. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito $dz = pdx + qdy$ fiat $pp+qq=1$, indolem istius functionis z in genere investigare.*

SOLUTIO

Cum per reductionem fiat

$$z = px + qy - \int(xdp + ydq),$$

ut irrationalia evitemus, ponamus

$$p = \frac{1-rr}{1+rr} \quad \text{et} \quad q = \frac{2r}{1+rr},$$

siquidem hinc fit $pp+qq=1$. Erit autem

$$dp = \frac{-4rdr}{(1+rr)^2} \quad \text{et} \quad dq = \frac{2dr(1-rr)}{(1+rr)^2}$$

hincque fit

$$z = \frac{(1-rr)x + 2ry}{1+rr} + 2\int \frac{2xr\,dr - y\,dr(1-rr)}{(1+rr)^2};$$

quae forma integralis cum sit functio ipsius r, statuatur ea $= f:r$ eiusque differentiale $= drf':r$, ex quo obtinebimus

$$\frac{2xr - y(1-rr)}{(1+rr)^2} = f':r \quad \text{et} \quad z = \frac{(1-rr)x + 2ry}{1+rr} + 2f:r.$$

Unde si eliciamus

$$x = \frac{(1-rr)y}{2r} + \frac{(1+rr)^2}{2r} f':r,$$

erit

$$z = \frac{(1+rr)y}{2r} + \frac{1-r^4}{2r} f':r + 2f:r.$$

COROLLARIUM 1

90. Si irrationalitatem non pertimescamus, ob

$$q = \sqrt{(1-pp)} \quad \text{et} \quad dq = \frac{-p\,dp}{\sqrt{(1-pp)}}$$

erit

$$z = px + y\sqrt{(1-pp)} - \int dp\left(x - \frac{py}{\sqrt{(1-pp)}}\right).$$

Posito ergo $z = px + y\sqrt{(1-pp)} - f:p$ erit $x - \frac{py}{\sqrt{(1-pp)}} = f':p.$

COROLLARIUM 2

91. Solutio simplicissima sine dubio prodit sumendo $f:p = 0$; unde cum sit $x = \frac{py}{\sqrt{(1-pp)}}$, erit

$$p = \frac{x}{\sqrt{(xx+yy)}} \quad \text{et} \quad \sqrt{(1-pp)} = \frac{y}{\sqrt{(xx+yy)}}$$

hincque

$$z = \frac{xx+yy}{\sqrt{(xx+yy)}} = \sqrt{(xx+yy)}.$$

Ex quo valore fit

$$\left(\frac{dz}{dx}\right) = p = \frac{x}{\sqrt{(xx+yy)}} \quad \text{et} \quad \left(\frac{dz}{dy}\right) = q = \frac{y}{\sqrt{(xx+yy)}}$$

ideoque $pp + qq = 1$.

COROLLARIUM 3

92. Si ponamus $p = \sin.\varphi$, erit $q = \cos.\varphi$ hincque

$$z = x\sin.\varphi + y\cos.\varphi - \int d\varphi(x\cos.\varphi - y\sin.\varphi);$$

erit hoc integrale $= f:\varphi$ eiusque differentiale $d\varphi f':\varphi$. Ex quo habebimus

$$z = x\sin.\varphi + y\cos.\varphi - f:\varphi \quad \text{et} \quad x\cos.\varphi - y\sin.\varphi = f':\varphi.$$

PROBLEMA 14

93. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito $dz = pdx + qdy$ quantitas q aequetur functioni datae ipsius p, indolem huius functionis z in genere investigare.*

SOLUTIO

Cum q sit functio data ipsius p, ponatur $dq = rdp$; erit etiam r functio data ipsius p. Aequatio ergo nostra generalis solutionem suppeditans induet hanc formam

$$z = px + qy - \int dp(x + ry),$$

unde patet integrale $\int dp(x + ry)$ fore functionem ipsius p; quae si generatim per $f:p$ exponatur eiusque differentiale per $dpf':p$, habebimus

$$z = px + qy - f:p \quad \text{et} \quad x + ry = f':p,$$

quae duae aequationes solutionem problematis universalissime complectuntur, siquidem $f:p$ functionem quamcunque ipsius p sive continuam sive discontinuam denotare potest.

COROLLARIUM 1

94. Cum sit q functio data ipsius p indeque $r = \frac{dq}{dp}$, si functio indefinita ipsius p ponatur $f:p = P$, ob $f':p = \frac{dP}{dp}$ solutio his aequationibus continebitur

$$z = px + qy - P \quad \text{et} \quad xdp + ydq = dP.$$

COROLLARIUM 2

95. Si ad constructionem utamur curva quacunque, in qua, si abscissa capiatur $= p$, applicata sit $= f' : p$, area eius curvae dabit valorem ipsius $f : p$. Sin autem applicata indicetur per $f : p$, tum $f' : p$ exprimet tangentem anguli, quem tangens curvae faciet cum axe.

SCHOLION

96. Duplici ergo modo curva quaecunque ad libitum descripta, sive sit continua seu aequatione quapiam analytica contenta sive libero manus ductu utcunque delineata, ad constructionem problematis adhiberi potest. Vel enim abscissa per p indicata applicata sumi potest ad $f : p$ vel ad $f' : p$ exprimendum nec facile dici potest, utrum ad praxin commodius sit futurum. Ubi autem huiusmodi problemata realia occurrunt, reliquae circumstantiae solutionem determinare solent, unde pro quovis casu constructio maxime idonea facile colligetur. Problemata autem mechanica hanc calculi integralis partem postulantia semper ad formulas differentiales secundi altiorumque ordinum deducunt, quarum resolutio ne suscipi quidem posse ante videtur, quam methodus pro formulis differentialibus primi gradus fuerit patefacta.

Hactenus quidem problemata proposita absolute resolvere licuit; nunc autem quando conditio praescripta relationem formularum $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ per reliquas variabiles x, y et z definit, negotium in genere non amplius succedit, nisi relatio praescripta unicam tantum variabilem cum binis formulis differentialibus coniungat.

CAPUT IV

DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER BINAS FORMULAS DIFFERENTIALES ET UNICAM TRIUM QUANTITATUM VARIABILIUM PROPONITUR

PROBLEMA 15

97. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito $dz = pdx + qdy$ sit $q = \frac{px}{a}$, indolem huius functionis in genere investigare.*

SOLUTIO

Cum sit

$$dz = pdx + \frac{pxdy}{a} = px\left(\frac{dx}{x} + \frac{dy}{a}\right)$$

haecque formula esse debeat integrabilis, necesse est, ut px ac proinde etiam z sit functio quantitatis $lx + \frac{y}{a}$. Quare solutio nostri problematis in genere ita se habebit, ut sit

$$z = f:\left(lx + \frac{y}{a}\right) \quad \text{et} \quad px = f':\left(lx + \frac{y}{a}\right)$$

sumendo scilicet perpetuo $d. f: u = duf': u$. Hinc autem erit

$$p = \frac{1}{x} f':\left(lx + \frac{y}{a}\right) \quad \text{et} \quad q = \frac{1}{a} f':\left(lx + \frac{y}{a}\right)$$

sicque $q = \frac{px}{a}$, omnino uti requiritur.

COROLLARIUM 1

98. Cum sit

$$z = px - \int x\,dp + \int \frac{px\,dy}{a} = px + \int px\left(\frac{dy}{a} - \frac{dp}{p}\right),$$

hinc alia solutio deduci potest. Si enim ponamus

$$\int px\left(\frac{dy}{a} - \frac{dp}{p}\right) = f:\left(\frac{y}{a} - lp\right),$$

erit $px = f':\left(\frac{y}{a} - lp\right)$ indeque

$$z = f':\left(\frac{y}{a} - lp\right) + f:\left(\frac{y}{a} - lp\right).$$

COROLLARIUM 2

99. Hac ergo solutione nova introducitur variabilis p, ex qua cum y coniuncta definitur primo

$$x = \frac{1}{p} f':\left(\frac{y}{a} - lp\right),$$

tum vero ipsa functio quaesita

$$z = px + f:\left(\frac{y}{a} - lp\right).$$

Huic autem solutioni praecedens sine dubio antecellit, cum illa quantitatem z immediate per x et y exprimat.

SCHOLION

100. Quo has duas solutiones inter se comparare queamus, quoniam functio arbitraria in utraque diversae est indolis, etiam charactere diverso utamur. Cum igitur prima praebeat

$$z = f:\left(\frac{y}{a} + lx\right) \quad \text{et} \quad px = f':\left(\frac{y}{a} + lx\right),$$

altera vero

$$z = F:\left(\frac{y}{a} - lp\right) + F':\left(\frac{y}{a} - lp\right) \quad \text{et} \quad px = F':\left(\frac{y}{a} - lp\right),$$

patet fore

$$f':\left(\frac{y}{a}+lx\right)=F':\left(\frac{y}{a}-lp\right) \quad \text{et} \quad f:\left(\frac{y}{a}+lx\right)=F:\left(\frac{y}{a}-lp\right)+F':\left(\frac{y}{a}-lp\right),$$

unde non solum relatio inter utriusque functionis f et F indolem definitur, sed etiam inde sequi debet fore

$$px=f':\left(\frac{y}{a}+lx\right),$$

id quod non parum videtur absconditum. Verum ob hoc ipsum istud problema eo magis est notatu dignum, quod solutio altera, qua nova variabilis p introducitur, congruit cum priore, ubi z per x et y immediate definitur, neque tamen consensus harum solutionum perspicue monstrari potest. Quamobrem quando ad eiusmodi solutiones pervenimus, uti in problematibus posterioribus capitis praecedentis usu venit, in quibus nova variabilis introducitur, non omnem statim spem eius eliminandae abiicere debemus, cum isto casu altera solutio ad priorem certe sit reductibilis, etiamsi methodus reducendi non perspiciatur, quam tamen infra § 119 exhibebimus.

PROBLEMA 16

101. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito $dz=pdx+qdy$ sit $q=pX+T$ existentibus X et T functionibus quibuscunque ipsius x, indolem istius functionis z in genere investigare.*

SOLUTIO

Cum ergo sit $dz=pdx+pXdy+Tdy$, statuatur $p=r-\frac{T}{X}$, ut prodeat

$$dz=rdx-\frac{Tdx}{X}+rXdy=\frac{-Tdx}{X}+rX\left(\frac{dx}{X}+dy\right),$$

qua reductione facta perspicuum est tam rX quam $\int rX\left(\frac{dx}{X}+dy\right)$ fore functionem quantitatis $y+\int\frac{dx}{X}$. Quare si ponamus

$$\int rX\left(\frac{dx}{X}+dy\right)=f:\left(y+\int\frac{dx}{X}\right),$$

erit

$$rX = f'\colon\left(y + \int\frac{dx}{X}\right)$$

ac tum functio quaesita erit

$$z = -\int\frac{Tdx}{X} + f\colon\left(y + \int\frac{dx}{X}\right),$$

quae ob functionem indefinitam $f\colon$ est completa. Tum vero erit

$$p = \frac{-T}{X} + \frac{1}{X}f'\colon\left(y + \int\frac{dx}{X}\right) \quad \text{et} \quad q = f'\colon\left(y + \int\frac{dx}{X}\right),$$

unde patet fore utique $q = pX + T$. Quoniam vero X et T sunt functiones datae ipsius x, formulae integrales $\int\frac{dx}{X}$ et $\int\frac{Tdx}{X}$ solutionem non turbant.

COROLLARIUM 1

102. Solutio aliquanto facilior redditur sumendo ex conditione praescripta $p = \frac{q}{X} - \frac{T}{X}$, unde fit

$$dz = -\frac{Tdx}{X} + \frac{qdx}{X} + qdy \quad \text{et} \quad z = -\int\frac{Tdx}{X} + \int q\left(dy + \frac{dx}{X}\right).$$

Iam manifesto est

$$\int q\left(dy + \frac{dx}{X}\right) = f\colon\left(y + \int\frac{dx}{X}\right)$$

sicque ipsa solutio praecedens resultat.

COROLLARIUM 2

103. Eodem modo resolvitur problema, si proponatur conditio $q = pY + V$ existentibus Y et V functionibus datis ipsius y. Tum enim erit

$$dz = pdx + pYdy + Vdy \quad \text{et} \quad z = \int Vdy + \int p(dx + Ydy).$$

Hic ergo fit

$$\int p(dx + Ydy) = f\colon(x + \int Ydy)$$

et solutio erit

$$z = \int Vdy + f\colon(x + \int Ydy),$$

unde fit

$$p = f'\colon(x + \int Ydy) \quad \text{et} \quad q = V + Yf'\colon(x + \int Ydy).$$

SCHOLION

104. Ex forma solutionis hic inventae discere poterimus, quomodo problema comparatum esse debeat, ut eius solutio hac ratione perfici et functio z per binas variabiles x et y exhiberi queat. Sint enim K et V functiones quaecunque ipsarum x et y indeque differentiando

$$dK = Ldx + Mdy \quad \text{et} \quad dV = Pdx + Qdy;$$

iam a solutione incipiamus ponamusque

$$z = K + f:V$$

eritque differentiando

$$dz = Ldx + Mdy + (Pdx + Qdy)f':V.$$

Cum iam hanc formam cum assumta

$$dz = pdx + qdy$$

comparando sit

$$p = L + Pf':V \quad \text{et} \quad q = M + Qf':V,$$

erit

$$Qp - Pq = LQ - MP.$$

Quare si hoc problema proponatur, ut posito $dz = pdx + qdy$ fieri debeat

$$q = \frac{Q}{P}p + M - \frac{LQ}{P},$$

solutio erit $z = K + f:V$, dummodo M et L itemque P et Q ita sint comparatae, ut sit

$$Ldx + Mdy = dK \quad \text{et} \quad Pdx + Qdy = dV;$$

verum hi casus ad sequens caput sunt referendi.[1])

PROBLEMA 17

105. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito $dz = pdx + qdy$ sit $q = Px + \Pi$ existentibus P et Π functionibus datis ipsius p, indolem istius functionis z in genere investigare.*

1) Vide praesertim § 146. F. E.

SOLUTIO

Cum igitur sit $dz = pdx + Pxdy + \Pi dy$, erit

$$z = px + \int (Pxdy + \Pi dy - xdp).$$

Statuatur $Px + \Pi = v$, ut sit $x = \frac{v - \Pi}{P}$, fietque

$$z = px + \int \left(vdy - \frac{vdp}{P} + \frac{\Pi dp}{P}\right).$$

Quare cum P et Π sint functiones ipsius p ideoque formula $\int \frac{\Pi dp}{P}$ data, habebitur

$$z = px + \int \frac{\Pi dp}{P} + \int v\left(dy - \frac{dp}{P}\right),$$

unde patet tam v quam $\int v\left(dy - \frac{dp}{P}\right)$ functionem esse debere formulae $y - \int \frac{dp}{P}$. Ponamus ergo

$$\int v\left(dy - \frac{dp}{P}\right) = f:\left(y - \int \frac{dp}{P}\right)$$

eritque

$$v = Px + \Pi = f':\left(y - \int \frac{dp}{P}\right)$$

et hinc

$$x = \frac{-\Pi}{P} + \frac{1}{P} f':\left(y - \int \frac{dp}{P}\right),$$

tum vero

$$z = \int \frac{\Pi dp}{P} - \frac{\Pi p}{P} + \frac{p}{P} f':\left(y - \int \frac{dp}{P}\right) + f:\left(y - \int \frac{dp}{P}\right).$$

COROLLARIUM 1

106. In solutione huius problematis iterum nova variabilis p introducitur, ex qua cum y coniunctim primo variabilis x, tum vero ipsa functio quaesita z determinatur.

COROLLARIUM 2

107. Neque vero hinc istam novam variabilem p ex calculo elidere licet, uti ante usu venit, propterea quod hic P et Π functiones ipsius p denotant, quarum indoles iam in ipsum problema ingreditur.

COROLLARIUM 3

108. Simili modo problema resolvetur, si permutandis x et y quantitas p ita per y et q detur, ut sit $p = Qy + \Xi$ denotantibus Q et Ξ functiones datas ipsius q.

SCHOLION

109. In hoc capite constituimus eiusmodi problemata tractare, quorum conditio aequatione inter binas formulas differentiales $\left(\frac{dz}{dx}\right) = p$, $\left(\frac{dz}{dy}\right) = q$ et unam ex tribus variabilibus x, y et z utcunque exprimitur. Problemata autem bina evoluta ex hoc genere certos casus complectuntur, quorum solutio peculiari methodo expediri potest simulque ad formulas simpliciores perducitur. In posteriori quidem relationem inter p, q et x ita assumsimus, ut sit $q = Px + \Pi$ seu ut in valore ipsius q per p et x expresso quantitas x unam dimensionem non excedat; in priori vero ita, ut sit $q = pX + T$ seu ut in valore ipsius q per p et x expresso quantitas p unicam obtineat dimensionem.

In genere autem notasse iuvabit tam quantitates p et x quam q et y inter se esse permutabiles. Cum enim sit $\int pdx = px - \int xdp$, loco

$$z = \int (pdx + qdy)$$

erit

$$z = px + \int (qdy - xdp).$$

Simili modo est

$$z = qy + \int (pdx - ydq),$$

tum vero etiam

$$z = px + qy - \int (xdp + ydq).$$

Quibus ergo casibus una harum quatuor formularum integralium redditur integrabilis, iisdem ternae reliquae etiam integrationem admittent. Cum igitur in superiori capite primam formulam resolverimus, si p vel q quomodocunque detur per x et y, ita eodem modo resolvetur formula secunda, si q per p et y, tertia autem, si p per x et q, at quarta, si vel x per p et q vel y per p et q utcunque datur; quae quaestiones cum generaliter expediri queant, eas in sequenti problemate evolvamus.

PROBLEMA 18

110. *Posito $dz = pdx + qdy$ si relatio inter p, q et x aequatione quacunque definiatur, indolem functionis z, quemadmodum ex binis variabilibus x et y determinetur, in genere investigare.*

SOLUTIO

Ex aequatione inter p, q et x proposita quaeratur valor ipsius x, qui functioni cuipiam ipsarum p et q aequabitur. Cum iam sit

$$z = px + qy - \int (xdp + ydq),$$

quoniam x est functio data ipsarum p et q, formula xdp integretur sumta quantitate q constante sitque

$$\int xdp = V + f:q$$

et erit V functio cognita ipsarum p et q, qua differentiata prodeat

$$dV = xdp + Sdq,$$

ubi S quoque erit functio data ipsarum p et q. Quia iam forma $\int (xdp + ydq)$ integrationem admittere debet, aequabitur formae $V + f:q$, unde differentiando concluditur

$$xdp + ydq = xdp + Sdq + dqf':q$$

ideoque

$$y = S + f':q \quad \text{et} \quad z = px + qy - V - f:q$$

seu

$$z = px + Sq + qf':q - f:q - V.$$

Solutio ergo ita se habet:

Primo ex conditione praescripta datur x per p et q; tum sumta q constante sit $V = \int xdp$ et vicissim $dV = xdp + Sdq$; inventis autem V et S per p et q reliquae quantitates y et z ita per easdem exprimentur, ut sit

$$y = S + f':q \quad \text{et} \quad z = px + Sq + qf':q - f:q - V,$$

quae solutio, quia $f:q$ functionem quamcunque ipsius q sive continuam sive discontinuam denotat, utique pro completa latissimeque patente est habenda.

ALITER

111. Vel ex aequatione inter p, q et x data quaeratur valor ipsius p per x et q expressus, ita ut p aequetur functioni cuipiam datae binarum variabilium x et q, per quas etiam reliquas quantitates y et z definire conemur. Ad hoc utamur formula

$$z = qy + \int (pdx - ydq),$$

et quia p est functio ipsarum x et q, dabitur earundem eiusmodi functio V, ut sit

$$dV = pdx + Rdq.$$

Statuatur ergo

$$\int (pdx - ydq) = V + f:q$$

eritque

$$y = -R - f':q \quad \text{et} \quad z = qy + V + f:q.$$

COROLLARIUM 1

112. Utraque solutio aeque commode adhiberi potest, si ex relatione inter p, q et x proposita tam quantitatem x quam p aeque commode definire liceat. Sin autem earum altera commodius definiri queat, ea solutione, quae ad istum casum est accommodata, erit utendum.

COROLLARIUM 2

113. Sin autem neque p neque x commode elici queat, tum nihilo minus hic resolutio aequationum cuiusque ordinis quin etiam transcendentium tanquam concessa assumitur. Ceterum etiamsi q facile per p et x definiatur, hinc calculus nihil iuvatur.

COROLLARIUM 3

114. Ex hoc problemate utpote latissime patente etiam bina praecedentia resolvi possunt; solutio autem hinc inventa a praecedente discrepabit, cum illa ex methodo particulari sit deducta. Operae autem pretium erit has duplices solutiones inter se comparare.

EXEMPLUM 1

115. *Si fuerit $q = pX + T$ existentibus X et T functionibus ipsius x, indolem functionis z investigare.*

Hic solutione utendum est posteriori, pro qua est $p = \frac{q-T}{X}$; nunc posita q constante prodit

$$V = \int p\,dx = q\int\frac{dx}{X} - \int\frac{T\,dx}{X}$$

hincque

$$R = \left(\frac{dV}{dq}\right) = \int\frac{dx}{X},$$

unde solutio his formulis continetur

$$q = pX + T, \quad y = -\int\frac{dx}{X} - f' : q, \quad z = -\int\frac{T\,dx}{X} - qf' : q + f : q;$$

solutio autem superior [§ 101] ita se habebat

$$q = pX + T, \quad q = f' : \left(y + \int\frac{dx}{X}\right) \quad \text{et} \quad z = -\int\frac{T\,dx}{X} + f : \left(y + \int\frac{dx}{X}\right).$$

SCHOLION

116. Consensus harum duarum solutionum ita ostendi potest, ut ex ea, quam hic invenimus, antecedens per legitimam consequentiam formetur.

Cum enim sit

$$f' : q = -y - \int\frac{dx}{X},$$

statuatur brevitatis gratia $y + \int\frac{dx}{X} = v$, ut sit $f' : q = -v$; erit ergo vicissim q aequalis functioni cuidam ipsius v, quae ponatur $q = F' : v$, unde fit $dq = dvF'' : v$, ergo

$$dqf' : q = -v\,dvF'' : v = -v\,d.F' : v,$$

ergo integrando

$$f : q = -\int v\,d.F' : v = -vF' : v + \int dvF' : v = -vF' : v + F : v.$$

Quare cum sit

$$z = -\int\frac{T\,dx}{X} - qf' : q + f : q,$$

erit

$$z = -\int \frac{T\,dx}{X} + vF':v - vF':v + F:v \quad \text{seu} \quad z = -\int \frac{T\,dx}{X} + F:\left(y + \int \frac{dx}{X}\right),$$

quae est ipsa solutio praecedens.

EXEMPLUM 2

117. *Si fuerit* $q = Px + \Pi$ *existentibus* P *et* Π *functionibus datis ipsius* p, *indolem functionis* z, *ut sit* $dz = pdx + qdy$, *investigare.*

Hic solutione priori utendum, cum sit $x = \frac{q-\Pi}{P}$. Sumto ergo q constante quaeratur

$$V = \int x\,dp = q\int \frac{dp}{P} - \int \frac{\Pi\,dp}{P},$$

unde fit

$$S = \left(\frac{dV}{dq}\right) = \int \frac{dp}{P}.$$

Solutio ergo praebet

$$y = \int \frac{dp}{P} + f':q$$

et

$$z = \frac{pq}{P} - \frac{p\Pi}{P} + q\int \frac{dp}{P} + qf':q - f:q - q\int \frac{dp}{P} + \int \frac{\Pi dp}{P}$$

sive

$$z = \frac{p(q-\Pi)}{P} + \int \frac{\Pi dp}{P} + qf':q - f:q.$$

Solutio autem eiusdem casus supra (§ 105) inventa erat

$$x = \frac{-\Pi}{P} + \frac{1}{P}f':\left(y - \int \frac{dp}{P}\right) \quad \text{et} \quad q = Px + \Pi$$

atque

$$z = \frac{-p\Pi}{P} + \int \frac{\Pi dp}{P} + \frac{p}{P}f':\left(y - \int \frac{dp}{P}\right) + f:\left(y - \int \frac{dp}{P}\right).$$

SCHOLION 1

118. Videamus, quomodo solutio hic inventa ad superiorem reduci queat. Cum ibi invenerimus

$$y - \int \frac{dp}{P} = f':q,$$

vicissim q aequabitur functioni quantitatis $y-\int\frac{dp}{P}$; ponatur ergo

$$q=F':\left(y-\int\frac{dp}{P}\right)$$

eritque statim

$$x=\frac{-\Pi}{P}+\frac{1}{P}F':\left(y-\int\frac{dp}{P}\right).$$

Sit brevitatis gratia $y-\int\frac{dp}{P}=v$, ut fiat $q=F':v$ et $v=f':q$; erit

$$F:v=\int q\,dv=qv-\int v\,dq=qv-\int dq f':q.$$

Ergo $F:v=qv-f:q$, ita ut sit

$$f:q=q\left(y-\int\frac{dp}{P}\right)-F:\left(y-\int\frac{dp}{P}\right)$$

seu

$$f:q=\left(y-\int\frac{dp}{P}\right)F':\left(y-\int\frac{dp}{P}\right)-F:\left(y-\int\frac{dp}{P}\right).$$

Quibus valoribus substitutis habebimus

$$x=\frac{-\Pi}{P}+\frac{1}{P}F':\left(y-\int\frac{dp}{P}\right)$$

et

$$z=\frac{-p\Pi}{P}+\frac{p}{P}F':\left(y-\int\frac{dp}{P}\right)+\int\frac{\Pi\,dp}{P}+\left(y-\int\frac{dp}{P}\right)F':\left(y-\int\frac{dp}{P}\right)$$
$$-\left(y-\int\frac{dp}{P}\right)F':\left(y-\int\frac{dp}{P}\right)+F:\left(y-\int\frac{dp}{P}\right)$$

seu

$$z=\frac{-p\Pi}{P}+\frac{p}{P}F':\left(y-\int\frac{dp}{P}\right)+\int\frac{\Pi\,dp}{P}+F:\left(y-\int\frac{dp}{P}\right),$$

quae est ipsa solutio ante inventa.

SCHOLION 2

119. Hoc consensu ostenso etiam consensum supra observatum (§ 100) demonstrare poterimus, qui multo magis absconditus videtur.

Altera autem solutio ibi inventa erat

$$px=F':\left(\frac{y}{a}-lp\right)\quad\text{et}\quad z=px+F:\left(\frac{y}{a}-lp\right),$$

ex quarum formula priori patet fore vicissim $\frac{y}{a} - lp$ functionem ipsius px. Hinc etiam $\frac{y}{a} - lp + lpx$ seu $\frac{y}{a} + lx$ aequabitur functioni ipsius px; denuo ergo vicissim px aequabitur functioni cuipiam ipsius $\frac{y}{a} + lx$. Ponatur ergo $px = f' : \left(\frac{y}{a} + lx\right)$, et cum sit

$$d.F : \left(\frac{y}{a} - lp\right) = \left(\frac{dy}{a} - \frac{dp}{p}\right) F' : \left(\frac{y}{a} - lp\right),$$

erit

$$F : \left(\frac{y}{a} - lp\right) = \int px\left(\frac{dy}{a} - \frac{dp}{p}\right) = \int px\left(\frac{dy}{a} + \frac{dx}{x}\right) - \int px\left(\frac{dx}{x} + \frac{dp}{p}\right)$$
$$= \int px\left(\frac{dy}{a} + \frac{dx}{x}\right) - px.$$

Iam pro px substituto valore $f' : \left(\frac{y}{a} + lx\right)$ obtinebitur

$$F : \left(\frac{y}{a} - lp\right) = -px + \int\left(\frac{dy}{a} + \frac{dx}{x}\right) f' : \left(\frac{y}{a} + lx\right) = -px + f : \left(\frac{y}{a} + lx\right),$$

ita ut hinc fiat $z = f : \left(\frac{y}{a} + lx\right)$, quae est ipsa solutio altera.

Hac igitur reductione haud parum luminis accenditur ad alia mysteria huius generis investiganda. Summa autem huius ratiocinii huc redit, ut si fuerit $r = f' : s$, fore etiam $r = F' : (s + R)$ denotante R functionem ipsius r, quod quidem per se est evidens, quia utrinque r per s determinatur. Cum ergo sit

$$f' : s = r = F' : (s + R),$$

erit

$$f : s = \int ds f' : s = \int r ds = \int r(ds + dR - dR) = \int (ds + dR) F' : (s + R) - \int r dR$$

ideoque

$$f : s = F : (s + R) - \int r dR,$$

unde loco functionum quantitatis s functiones quantitatis $s + R$ introduci possunt. Scilicet si sit $r = f' : s$, sumi potest $r = F' : (s + R)$ existente R functione quacunque ipsius r, tum vero erit

$$f : s = F : (s + R) - \int r dR.$$

EXEMPLUM 3

120. *Posito $dz = pdx + qdy$ si x aequetur functioni homogeneae n dimensionum ipsarum p et q, indolem functionis z investigare.*

Cum x detur per p et q, utendum erit solutione priori et ob $x =$ functioni homogeneae n dimensionum ipsarum p et q ponatur $p = qr$ fietque $x = q^n R$ existente R functione ipsius r tantum. Sumatur nunc q constans et quaeratur

$$V = \int x dp = \int q^{n+1} R dr$$

ob $dp = qdr$ eritque

$$V = q^{n+1} \int R dr,$$

quod integrale datur. Hinc differentiando erit

$$dV = q^{n+1} R dr + (n+1) q^n dq \int R dr;$$

quae ut cum

$$dV = x dp + S dq = q^n R dp + S dq$$

comparari possit, quia ob $dp = qdr + rdq$ est

$$dV = q^{n+1} R dr + q^n R r dq + S dq,$$

erit

$$S = -q^n R r + (n+1) q^n \int R dr,$$

unde fit

$$y = -q^n R r + (n+1) q^n \int R dr + f' : q \quad \text{et} \quad x = q^n R$$

atque

$$z = n q^{n+1} \int R dr + q f' : q - f : q$$

existente $p = qr$.

COROLLARIUM 1

121. Sit $x = \frac{p^m}{q^m}$ et posito $p = qr$ erit $x = r^m$ ideoque $n = 0$ et $R = r^m$, unde fit

$$y = -r^{m+1} + \frac{r^{m+1}}{m+1} + f' : q = \frac{-m}{m+1} r^{m+1} + f' : q \quad \text{et} \quad z = q f' : q - f : q.$$

Quare ob $r = x^{\frac{1}{m}}$ erit

$$y = \frac{-m}{m+1} x^{\frac{m+1}{m}} + f' : q.$$

COROLLARIUM 2

122. Eodem casu ergo, quo $x = \frac{p^m}{q^m}$, aequabitur q functioni quantitatis $y + \frac{m}{m+1} x^{\frac{m+1}{m}}$; quae quantitas si ponatur $= v$ et $q = F' : v$, ut sit $v = f' : q$, erit

$$f : q = \int dq f' : q = \int v dv F'' : v$$

ob $dq = dv F'' : v$, unde concluditur

$$f : q = vF' : v - F : v \quad \text{et} \quad z = F : v = F : \left(y + \frac{m}{m+1} x^{\frac{m+1}{m}}\right).$$

EXEMPLUM 4

123. *Duarum variabilium x et y eiusmodi functionem z investigare, ut posito $dz = pdx + qdy$ fiat $p^3 + x^3 = 3pqx$.*

Consideretur forma

$$z = qy + \int (pdx - ydq),$$

ubi iam formulam $pdx - ydq$ integrabilem reddi oportet. Statuatur $p = ux$ et conditio praescripta dat $x(1 + u^3) = 3qu$; unde fit

$$x = \frac{3qu}{1+u^3} \quad \text{et} \quad p = \frac{3quu}{1+u^3},$$

tum vero

$$dx = \frac{3qdu(1-2u^3)}{(1+u^3)^2} + \frac{3udq}{1+u^3}$$

sicque habebitur

$$z = qy + \int \left(\frac{9qquudu(1-2u^3)}{(1+u^3)^3} + \frac{9qu^3dq}{(1+u^3)^2} - ydq\right);$$

at

$$\int \frac{9qquudu(1-2u^3)}{(1+u^3)^3} = \frac{3qq(1+4u^3)}{2(1+u^3)^2} - \int \frac{3q(1+4u^3)dq}{(1+u^3)^2},$$

ergo

$$z = qy + \frac{3qq(1+4u^3)}{2(1+u^3)^2} - \int dq \left(y + \frac{3q}{1+u^3}\right).$$

Quare necesse est esse $y + \frac{3q}{1+u^3}$ functionem ipsius q tantum, quae sit

$= -f':q$, unde fit

$$y = -\frac{3q}{1+u^3} - f':q \quad \text{et} \quad z = qy + \frac{3qq(1+4u^3)}{2(1+u^3)^2} + f:q$$

seu

$$z = \frac{3qq(2u^3-1)}{2(1+u^3)^2} - qf':q + f:q$$

existente $x = \frac{3qu}{1+u^3}$. Ex quibus tribus aequationibus si eliminentur binae quantitates q et u, orietur aequatio inter z et x, y, quae quaeritur.

COROLLARIUM 1

124. Ex aequatione pro y inventa colligitur

$$\frac{3}{1+u^3} = \frac{-y-f':q}{q},$$

aequatio autem pro z inventa abit in hanc

$$z = \frac{3qq}{1+u^3} - \frac{9qq}{2(1+u^3)^2} - qf':q + f:q,$$

quae eliso u transmutatur in hanc

$$z = -qy - 2qf':q - \frac{1}{2}(y+f':q)^2 + f:q;$$

tum vero est

$$x = -u(y+f':q),$$

unde reperitur $u = \frac{-x}{y+f':q}$ hincque

$$x^3 = 3q(y+f':q)^2 + (y+f':q)^3.$$

COROLLARIUM 2

125. Si sumamus $f':q = a$, erit $f:q = aq + b$ et postrema aequatio praebet $q = \frac{x^3-(y+a)^3}{3(y+a)^2}$. Cum deinde pro hoc casu fiat

$$z = -qy - aq - \frac{1}{2}(y+a)^2 + b,$$

proveniet loco q valorem inventum substituendo

$$z = \frac{6b(y+a) - (y+a)^3 - 2x^3}{6(y+a)}.$$

COROLLARIUM 3

126. Cum in genere sit

$$x^3 = (y + f' : q)^2 (y + 3q + f' : q),$$

ponamus $f' : q = a - 3q$ ideoque $f : q = b + aq - \frac{3}{2} qq$, ut fiat

$$(y + a - 3q)^2 = \frac{x^3}{y + a},$$

eritque

$$y + a - 3q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} \quad \text{et} \quad q = \frac{1}{3}(y + a) - \frac{x\sqrt{x}}{3\sqrt{(y+a)}}.$$

Hinc ergo prodit

$$f' : q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} - y$$

et

$$f : q = b + \frac{a(y+a)}{3} - \frac{ax\sqrt{x}}{3\sqrt{(y+a)}} - \frac{1}{6}(y + a)^2 + \frac{1}{3} x\sqrt{x(y + a)} - \frac{x^3}{6(y+a)}$$

$$= b + \frac{aa - yy}{6} + \frac{xy\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{6(y+a)}$$

atque

$$z = -\frac{1}{3} y(y + a) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - 2aq + 6qq - \frac{x^3}{2(y+a)} + b + aq - \frac{3}{2} qq$$

seu

$$z = b - \frac{1}{3} y(y + a) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{2(y+a)} - aq + \frac{9}{2} qq$$

et facta reductione

$$z = b + \frac{1}{6}(y + a)^2 - \frac{2}{3} x\sqrt{x(y + a)}.$$

COROLLARIUM 4

127. Quodsi hic sumatur $a = 0$ et $b = 0$, erit per expressionem satis simplicem

$$z = \frac{1}{6} yy - \frac{2}{3} x\sqrt{xy},$$

quae quomodo conditioni praescriptae satisfaciat, ita apparet. Per differentiationem colligitur

$$p = \left(\frac{dz}{dx}\right) = -\sqrt{xy} \quad \text{et} \quad q = \left(\frac{dz}{dy}\right) = \frac{1}{3}y - \frac{x\sqrt{x}}{3\sqrt{y}}$$

hincque

$$p^3 + x^3 = -xy\sqrt{xy} + x^3,$$

at $3pq = xx - y\sqrt{xy}$ ideoque $3pqx = x^3 - xy\sqrt{xy}$, ergo

$$p^3 + x^3 = 3pqx.$$

SCHOLION

128. Successit ergo solutio, quando aequatio quaecunque inter p, q et x proponitur, etiamsi casibus, quibus inde neque x neque p elici potest, difficultas quaedam restat, quae autem resolutionem aequationum finitarum potissimum afficit, quam hic merito concedi postulamus. Interim ex postremo exemplo perspicitur, quomodo operatio sit instituenda, si ope substitutionis idoneae aequatio proposita ad resolutionem accommodari queat, cui autem negotio hic amplius non immoror. Neque etiam eos casus, quibus inter p, q et y relatio quaedam praescribitur, hic seorsim evolvam, cum ob permutabilitatem ipsarum x et y, qua etiam p et q permutantur, hi casus ad praecedentes sponte revocentur. Superest igitur casus, quo aequatio inter p, q et z proponitur, ubi quidem statim manifestum est in aequatione $dz = pdx + qdy$ quantitates p et q non uti functiones ipsarum x et y spectari posse, quoniam etiam a z pendent, neque ergo earum indoles inde determinari poterit, ut formula $pdx + qdy$ integrabilis evadat. Verum sine discrimine conditio ea est definienda, ut aequatio differentialis $dz - pdx - qdy = 0$ fiat possibilis; ad quod ex principiis supra stabilitis (§ 6) requiritur, ut posito

$$\left(\frac{dq}{dz}\right) = L, \quad -\left(\frac{dp}{dz}\right) = M \quad \text{et} \quad \left(\frac{dp}{dy}\right) - \left(\frac{dq}{dx}\right) = N$$

sit

$$Lp + Mq - N = 0 \quad \text{seu} \quad p\left(\frac{dq}{dz}\right) - q\left(\frac{dp}{dz}\right) + \left(\frac{dq}{dx}\right) - \left(\frac{dp}{dy}\right) = 0.$$

Quare proposita aequatione quacunque inter p, q et z eas conditiones in genere investigari oportet, ut huic requisito satisfiat.

PROBLEMA 19

129. *Si posito* $dz = pdx + qdy$ *debeat esse* $p + q = \frac{z}{a}$, *relationem functionis* z *ad variabiles* x *et* y *in genere investigare.*

SOLUTIO

Cum sit $q = \frac{z}{a} - p$, aequatio nostra hanc induet formam

$$dz = pdx - pdy + \frac{zdy}{a}$$

seu

$$p(dx - dy) = \frac{adz - zdy}{a} = z\left(\frac{dz}{z} - \frac{dy}{a}\right).$$

Quoniam igitur ambae formulae $dx - dy$ et $\frac{dz}{z} - \frac{dy}{a}$ per se sunt integrabiles, ob

$$\frac{dz}{z} - \frac{dy}{a} = \frac{p}{z}(dx - dy)$$

necesse est, ut $\frac{p}{z}$ sit functio quantitatis $x - y$; ponatur ergo $\frac{p}{z} = f' : (x - y)$, ut fiat

$$lz - \frac{y}{a} = f : (x - y).$$

Definiri ergo potest z per x et y, et cum sit $e^{f:(x-y)}$ etiam functio ipsius $x - y$, si ea ponatur $= F : (x - y)$, erit

$$z = e^{\frac{y}{a}} F : (x - y),$$

unde fit

$$\left(\frac{dz}{dx}\right) = p = e^{\frac{y}{a}} F' : (x - y) \quad \text{et} \quad \left(\frac{dz}{dy}\right) = q = -e^{\frac{y}{a}} F' : (x - y) + \frac{1}{a} e^{\frac{y}{a}} F : (x - y)$$

ideoque

$$p + q = \frac{1}{a} e^{\frac{y}{a}} F : (x - y) = \frac{z}{a},$$

uti requiritur.

COROLLARIUM 1

130. Ex hoc exemplo intelligitur, quomodo certa functio ipsarum p et q quantitati z aequari possit, etiamsi p et q sint functiones ipsarum x et y. Simul scilicet ratio integralis formulae $dz = pdx + qdy$ introducitur in calculum.

COROLLARIUM 2

131. Forma $e^{\frac{y}{a}} F:(x-y)$ pro valore ipsius z inventa per functionem quamvis ipsius $x-y$ multiplicari potest. Si ergo multiplicetur per $e^{\frac{x-y}{a}}$, fit $z = e^{\frac{x}{a}} F:(x-y)$. Sin autem multiplicetur per $e^{\frac{x-y}{2a}}$, fit $z = e^{\frac{x+y}{2a}} F:(x-y)$, quae formae problemati aeque satisfaciunt.

PROBLEMA 20

132. *Si posito $dz = pdx + qdy$ quantitas z aequari debeat functioni datae ipsarum p et q, indolem, qua z per x et y definitur, in genere investigare.*

SOLUTIO

Ex formula proposita habemus $dy = \frac{dz}{q} - \frac{pdx}{q}$; statuatur $p = qr$, ut sit z aequalis functioni ipsarum q et r, et ex $dy = \frac{dz}{q} - rdx$ elicitur

$$y = \frac{z}{q} - rx + \int \left(\frac{z\,dq}{qq} + xdr \right),$$

quam formulam integrabilem reddi oportet. Cum igitur z sit functio data ipsarum q et r, posito r constante quaeratur integrale formulae $\frac{z\,dq}{qq}$ sitque

$$\int \frac{z\,dq}{qq} = V + f:r,$$

unde differentiando prodeat

$$dV = \frac{z\,dq}{qq} + Rdr,$$

ac iam patet esse debere $x = R + f':r$ indeque obtineri

$$y = \frac{z}{q} - Rr - rf':r + V + f:r,$$

quibus duabus aequationibus relatio inter quantitates propositas determinatur.

Primo igitur posito $p = qr$ datur z per q et r. Deinde sumto r constante integretur formula $\frac{z\,dq}{qq}$ sitque integrale resultans $V = \int \frac{z\,dq}{qq}$, quod etiam per q et r datur, unde sumto q constante colligitur $R = \left(\frac{dV}{dr}\right)$. Quibus inventis erit

$$x = R + f' : r \quad \text{et} \quad y = \frac{z}{q} - rx + V + f : r$$

sicque omnes quantitates per binas variabiles q et r determinantur.

COROLLARIUM 1

133. Quia permutatis x et y litterae p et q permutantur, simili modo nostram investigationem incipere potuissemus ab aequatione $dx = \frac{dz}{p} - \frac{q\,dy}{p}$ similisque solutio prodiisset, quae quidem forma diversa, at re congruens esset.

COROLLARIUM 2

134. Iam scilicet posito $q = ps$, ut sit $dx = \frac{dz}{p} - s\,dy$, erit

$$x = \frac{z}{p} - sy + \int \left(\frac{z\,dp}{pp} + y\,ds\right).$$

Iam sumto s constante ponatur $\int \frac{z\,dp}{pp} = U$, quae quantitas per p et s determinatur, ex ea vero prodeat $\left(\frac{dU}{ds}\right) = S$; erit

$$y = S + f' : s \quad \text{et} \quad x = \frac{z}{p} - sy + U + f : s.$$

EXEMPLUM 1

135. *Si esse debeat $p + q = \frac{z}{a}$, solutionem pro hoc casu exhibere.*

Posito $p = qr$ erit $z = aq(1 + r)$; nunc sumto r constante erit

$$V = \int \frac{z\,dq}{qq} = a(1 + r)\,lq \quad \text{et} \quad R = \left(\frac{dV}{dr}\right) = a\,lq.$$

Hinc reperitur

$$x = alq + f' : r \quad \text{et} \quad y = \frac{z}{q} - arlq - rf' : r + a(1+r)lq + f : r$$

seu

$$y = a(1+r) + alq - rf' : r + f : r.$$

Si hinc q elidere velimus, ob $q = \frac{z}{a(1+r)}$ solutio his duabus aequationibus continetur

$$x = al\frac{z}{a(1+r)} + f' : r$$

et

$$y = al\frac{z}{a(1+r)} + a(1+r) - rf' : r + f : r.$$

Unde sequenti modo praecedens solutio [§ 129, 131] elici potest. Ex forma priori est

$$\frac{x}{a} - l\frac{z}{a} = -l(1+r) + \frac{1}{a}f' : r = \text{funct. } r,$$

ex ambabus vero

$$y - x = a(1+r) - (1+r)f' : r + f : r = \text{funct. } r.$$

Cum ergo tam $\frac{x}{a} - l\frac{z}{a}$ seu $ze^{-\frac{x}{a}}$ quam $y - x$ sit functio ipsius r, altera forma aequabitur functioni alterius, unde statui potest

$$ze^{-\frac{x}{a}} = F : (y - x) \quad \text{seu} \quad z = e^{\frac{x}{a}}F : (y - x),$$

quae est solutio ante inventa.

EXEMPLUM 2

136. *Si posito $dz = pdx + qdy$ debeat esse $z = apq$, relationem inter x, y et z investigare.*

Posito $p = qr$ erit $z = aqqr$ et sumto r constante fit $V = \int \frac{zdq}{qq} = aqr$ hincque $R = \left(\frac{dV}{dr}\right) = aq$. Quocirca habebimus

$$x = aq + f' : r \quad \text{et} \quad y = aqr - rf' : r + f : r$$

seu ob $r = \frac{z}{aqq}$ erit

$$x = aq + f' : \frac{z}{aqq} \quad \text{et} \quad y = \frac{z}{q} - \frac{z}{aqq}f' : \frac{z}{aqq} + f : \frac{z}{aqq}.$$

Hic in genere notemus, si sit $f':r=v$ ponamusque $r=F':v$, ob $dr=dvF'':v$ fore

$$f:r=\int drf':r=\int vdvF'':v=vF':v-F:v$$

seu $f:r=vF':v-F:v$ hincque $f:r-rf':r=-F:v$. Quare cum sit $f':r=x-aq$, si ponamus $r=F':(x-aq)$, erit

$$f:r-rf':r=-F:(x-aq) \quad \text{et} \quad y=aqF':(x-aq)-F:(x-aq)$$

atque

$$z=aqqF':(x-aq).$$

SCHOLION

137. Hae postremae formulae ita statim ex conditione quaestionis elici possunt. Nam ob $p=\frac{z}{aq}$ erit

$$dz=\frac{zdx}{aq}+qdy \quad \text{et} \quad dy=\frac{dz}{q}-\frac{zdx}{aqq}$$

hincque

$$y=\frac{z}{q}+\int\left(\frac{zdq}{qq}-\frac{zdx}{aqq}\right)=\frac{z}{q}+\int\frac{z}{qq}\left(dq-\frac{dx}{a}\right),$$

ubi manifestum est esse $\frac{z}{qq}$ functionem quantitatis $q-\frac{x}{a}$. Quare posito $\frac{z}{qq}=F':\left(q-\frac{x}{a}\right)$ erit

$$y=\frac{z}{q}+F:\left(q-\frac{x}{a}\right).$$

Quin etiam indidem alia solutio deduci potest ponendo

$$dx=\frac{aq}{z}(dz-qdy),$$

quae posito $z=qv$ abit in

$$dx=\frac{a}{v}(vdq+qdv-qdy),$$

unde

$$x=aq+\int\frac{aq}{v}(dv-dy).$$

Quare ponatur $\frac{aq}{v}=f':(v-y)$ eritque $x=aq+f:(v-y)$. Iam restituto valore $v=\frac{z}{q}$ habebitur

$$\frac{aqq}{z}=f':\left(\frac{z}{q}-y\right) \quad \text{et} \quad x-aq=f:\left(\frac{z}{q}-y\right).$$

Prima autem solutio ad eliminanda q et r est aptissima in exemplis. Si enim ponatur $f':r = \frac{b}{\sqrt{r}} + c$, erit $f:r = 2b\sqrt{r} + cr + d$; hinc

$$z = aqqr \quad \text{et} \quad x = aq + \frac{b}{\sqrt{r}} + c, \quad y = aqr + b\sqrt{r} + d.$$

Iam ob $r = \frac{z}{aqq}$ fit

$$x = aq + bq\sqrt{\frac{a}{z}} + c \quad \text{et} \quad y = \frac{z}{q} + \frac{b}{q}\sqrt{\frac{z}{a}} + d.$$

Hinc

$$x - c = q\left(a + \frac{b\sqrt{a}}{\sqrt{z}}\right) \quad \text{et} \quad y - d = \frac{z}{aq}\left(a + \frac{b\sqrt{a}}{\sqrt{z}}\right)$$

et multiplicando eliditur q fitque

$$(x - c)(y - d) = \frac{z}{a}\left(a + \frac{b\sqrt{a}}{\sqrt{z}}\right)^2 = (b + \sqrt{az})^2,$$

ita ut sit

$$b + \sqrt{az} = \sqrt{(x - c)(y - d)}$$

et proinde

$$z = \frac{(x-c)(y-d) - 2b\sqrt{(x-c)(y-d)} + bb}{a},$$

quae, si $b = c = d = 0$, dat casum simplicissimum $z = \frac{xy}{a}$.

CAPUT V

DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER QUANTITATES $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ ET BINAS TRIUM VARIABILIUM x, y, z QUAECUNQUE DATUR

PROBLEMA 21

138. *Si posito* $dz = pdx + qdy$ *debeat esse* $px + qy = 0$, *functionis* z *indolem per* x *et* y *in genere investigare.*

SOLUTIO

Cum sit $q = -\frac{px}{y}$, erit

$$dz = pdx - \frac{pxdy}{y} = px\left(\frac{dx}{x} - \frac{dy}{y}\right)$$

seu

$$dz = py\left(\frac{dx}{y} - \frac{xdy}{yy}\right) = pyd.\frac{x}{y}.$$

Unde patet py esse debere functionem ipsius $\frac{x}{y}$, ac si ponatur $py = f':\frac{x}{y}$, fore $z = f:\frac{x}{y}$. Perpetuo scilicet in designandis functionibus hac lege utemur, ut sit $d.f:v = dvf':v$ sicque porro $d.f':v = dvf'':v$ et $d.f'':v = dvf''':v$ etc. At $f:\frac{x}{y}$ denotat functionem quamcunque homogeneam ipsarum x et y nullius dimensionis, ac si z fuerit talis functio quaecunque et differentiando prodeat $dz = pdx + qdy$, semper erit $px + qy = 0$.

COROLLARIUM 1

139. Quodsi ergo z fuerit functio homogenea nullius dimensionis ipsarum x et y, ob $p=\left(\frac{dz}{dx}\right)$ et $q=\left(\frac{dz}{dy}\right)$ erit

$$x\left(\frac{dz}{dx}\right)+y\left(\frac{dz}{dy}\right)=0,$$

quam veritatem quidem iam supra[1]) elicuimus.

COROLLARIUM 2

140. Tum vero cum sit

$$p=\frac{1}{y}f':\frac{x}{y} \quad \text{et} \quad q=\frac{-x}{yy}f':\frac{x}{y},$$

erit p functio homogenea ipsarum x et y numeri dimensionum $=-1$, et si sit $q=\frac{-px}{y}$, ipsa functio z reperitur ex integratione $z=\int pyd.\frac{x}{y}$.

SCHOLION

141. Simili modo solvitur problema, si posito $dz=pdx+qdy$ fieri debeat $mpx+nqy=a$. Tum enim ob $q=\frac{a}{ny}-\frac{mpx}{ny}$ erit

$$dz=\frac{ady}{ny}+pdx-\frac{mpxdy}{ny}$$

seu

$$dz=\frac{ady}{ny}+\frac{px}{n}\left(\frac{ndx}{x}-\frac{mdy}{y}\right)=\frac{ady}{ny}+\frac{py^m}{nx^{n-1}}d.\frac{x^n}{y^m},$$

unde solutio praebet

$$\frac{py^m}{nx^{n-1}}=f':\frac{x^n}{y^m} \quad \text{et} \quad z=\frac{a}{n}ly+f:\frac{x^n}{y^m}.$$

Quin etiam hoc generalius problema resolvi potest, quo esse debet

$$pX+qY=A$$

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1) Vide *Institutionum calculi integralis* vol. I § 481, *Leonhardi Euleri Opera omnia*, series I, vol. 11, p. 301. F. E.

existente X functione ipsius x et Y ipsius y. Cum enim inde fiat $q = \frac{A}{Y} - \frac{pX}{Y}$, erit

$$dz = \frac{A\,dy}{Y} + p\,dx - \frac{pX\,dy}{Y} = \frac{A\,dy}{Y} + pX\left(\frac{dx}{X} - \frac{dy}{Y}\right).$$

Statui ergo debet

$$pX = f':\left(\int\frac{dx}{X} - \int\frac{dy}{Y}\right)$$

indeque fit

$$z = A\int\frac{dy}{Y} + f:\left(\int\frac{dx}{X} - \int\frac{dy}{Y}\right).$$

PROBLEMA 22

142. *Si posito $dz = p\,dx + q\,dy$ debeat esse $\frac{q}{p}$ aequale functioni datae cuicunque ipsarum x et y, indolem functionis z in genere investigare.*

SOLUTIO

Sit V ista functio data ipsarum x et y, ut sit $q = pV$, et habebitur $dz = p(dx + V\,dy)$. Dabitur iam multiplicator M, itidem functio ipsarum x et y, ut $M(dx + V\,dy)$ fiat integrabile. Ponatur ergo $M(dx + V\,dy) = dS$ ac dabitur etiam S, functio ipsarum x et y. Cum ergo sit $dz = \frac{p\,dS}{M}$, perspicuum est quantitatem $\frac{p}{M}$ aequari debere functioni ipsius S; quare si ponamus $\frac{p}{M} = f':S$, fiet $z = f:S$ indeque erit

$$p = Mf':S \quad \text{et} \quad q = MVf':S.$$

COROLLARIUM 1

143. Hoc ergo casu functio quaesita z statim invenitur per x et y expressa, quoniam S per x et y datur. Fieri autem potest, ut S prodeat quantitas transcendens, quin etiam ut per methodos adhuc cognitas multiplicator M ne inveniri quidem possit.

COROLLARIUM 2

144. Si V sit functio nullius dimensionis ipsarum x et y, erit $M = \frac{1}{x + Vy}$. Seu posito $x = vy$ fiet V functio ipsius v et

$$dS = M(y\,dv + v\,dy + V\,dy).$$

Capiatur $M = \frac{1}{y(v+V)}$ eritque $dS = \frac{dy}{y} + \frac{dv}{v+V}$, unde reperitur

$$z = f:\left(ly + \int \frac{dv}{v+V}\right).$$

SCHOLION

145. Ob permutabilitatem ipsarum p et x, item q et y simili modo sequentia problemata resolvi possunt.

I. Si debeat esse $q = xV$ existente V functione quacunque ipsarum p et y, consideretur forma

$$z = px + \int (qdy - xdp) = px + \int x(Vdy - dp).$$

Quaeratur multiplicator M, ut sit

$$M(Vdy - dp) = dS;$$

erit S functio ipsarum p et y atque

$$z = px + \int \frac{xdS}{M},$$

ex quo colligitur haec solutio

$$\frac{x}{M} = f':S \quad \text{et} \quad z = pMf':S + f:S.$$

II. Si debeat esse $y = pV$ existente V functione quacunque ipsarum x et q, consideretur forma

$$z = qy + \int (pdx - ydq) = qy + \int p(dx - Vdq).$$

Quaeratur multiplicator M, ut sit

$$M(dx - Vdq) = dS;$$

erit S functio ipsarum x et q et

$$z = qy + \int \frac{pdS}{M}.$$

Quare fit

$$\frac{p}{M} = f':S \quad \text{et} \quad z = qy + f:S$$

seu ob $p = \frac{y}{V}$ erit

$$y = MVf':S \quad \text{et} \quad z = qMVf':S + f:S.$$

III. Si debeat esse $y = xV$ existente V functione quacunque ipsarum p et q, consideretur haec forma

$$z = px + qy - \int (xdp + xVdq).$$

Quaeratur multiplicator M, ut fiat

$$M(dp + Vdq) = dS;$$

erit S functio ipsarum p et q et

$$z = px + qy - \int \frac{xdS}{M},$$

unde haec solutio nascitur

$$\frac{x}{M} = f' : S \quad \text{et} \quad z = px + qy - f : S.$$

Omnes hi casus huc redeunt, ut quaternarum quantitatum p, x, q, y vel $\frac{q}{p}$ vel $\frac{q}{x}$ vel $\frac{y}{p}$ vel $\frac{y}{x}$ aequetur functioni cuicunque binarum reliquarum.

PROBLEMA 23

146. *Si posito $dz = pdx + qdy$ requiratur, ut sit $q = pV + U$ existente tam V quam U functione quacunque binarum variabilium x et y, indolem functionis z in genere investigare.*

SOLUTIO

Cum ob $q = pV + U$ sit

$$dz = p(dx + Vdy) + Udy,$$

quaeratur primo multiplicator M formulam $dx + Vdy$ reddens integrabilem sitque

$$M(dx + Vdy) = dS;$$

erunt M et S functiones ipsarum x et y fietque

$$dz = \frac{pdS}{M} + Udy.$$

Cum iam sit S functio ipsarum x et y, inde x per y et S definiri potest, quo valore introducto fient U et M functiones ipsarum y et S. Nunc sumto S constante integretur formula Udy sitque

$$\int Udy = T + f:S$$

ac posito

$$dT = Udy + WdS$$

fiet

$$\frac{p}{M} = W + f':S \quad \text{et} \quad z = T + f:S$$

sicque omnia per binas variabiles y et S exprimentur.

COROLLARIUM 1

147. Datis ergo binarum variabilium x et y functionibus V et U, ut sit $q = pV + U$, solutio problematis primo postulat, ut multiplicator M investigetur formulam $dx + Vdy$ integrabilem reddens, quo invento habebitur functio S earundem variabilium x et y, ut sit

$$S = \int M(dx + Vdy).$$

COROLLARIUM 2

148. In hunc finem considerari conveniet aequationem differentialem $dx + Vdy = 0$; haec enim si integrari poterit, simul inde colligi potest multiplicator M, ut formula $M(dx + Vdy)$ fiat verum differentiale cuiusdam functionis S, quae propterea hinc invenietur.

COROLLARIUM 3

149. Inventa porro hac functione S quantitas x per y et S exprimi debet, ita ut x aequetur functioni ipsarum y et S; quo valore in quantitate U substituto quaeratur integrale $\int Udy = T$ spectata S ut constante sicque obtinebitur T functio ipsarum y et S.

COROLLARIUM 4

150. Denique inventa hac functione T sit $W = \left(\frac{dT}{dS}\right)$, unde tandem colligitur solutio problematis his duabus formulis contenta

$$\frac{p}{M} = W + f' : S \quad \text{et} \quad z = T + f : S;$$

ubi cum S sit functio ipsarum x et y, pro z statim reperitur functio ipsarum x et y.

COROLLARIUM 5

151. Si U sit functio ipsius y tantum, non opus est illa expressione ipsius x per y et S, sed $T = \int U dy$ erit quoque functio ipsius y tantum, hinc $W = \left(\frac{dT}{dS}\right) = 0$. Hic autem casus manifesto reducitur ad praecedentem [§ 142] ponendo z loco $z - \int U dy$.

EXEMPLUM 1

152. *Si posito* $dz = pdx + qdy$ *debeat esse* $q = \frac{px}{y} + \frac{y}{x}$, *indolem functionis* z *investigare.*

Hic ergo est

$$V = \frac{x}{y} \quad \text{et} \quad U = \frac{y}{x},$$

unde ob

$$dx + Vdy = dx + \frac{xdy}{y}$$

erit multiplicator $M = y$ et $dS = ydx + xdy$, hinc $S = xy$ sicque habebitur

$$x = \frac{S}{y} \quad \text{et} \quad U = \frac{yy}{S}.$$

Iam erit

$$T = \int U dy = \int \frac{yydy}{S} = \frac{y^3}{3S} \quad \text{et} \quad W = \frac{-y^3}{3SS}.$$

Quare pro solutione huius exempli habebimus

$$\frac{p}{y} = \frac{-y^3}{3SS} + f' : S \quad \text{et} \quad z = \frac{y^3}{3S} + f : S$$

seu ob $S = xy$ erit

$$z = \frac{yy}{3x} + f : xy.$$

EXEMPLUM 2

153. *Si posito $dz = pdx + qdy$ debeat esse $px + qy = n\sqrt{(xx+yy)}$, indolem functionis z investigare.*

Cum hic sit $q = \frac{-px}{y} + \frac{n}{y}\sqrt{(xx+yy)}$, erit

$$V = \frac{-x}{y} \quad \text{et} \quad U = \frac{n}{y}\sqrt{(xx+yy)},$$

ergo $dS = M\left(dx - \frac{xdy}{y}\right)$, quare capiatur $M = \frac{1}{y}$, ut fiat $dS = \frac{dx}{y} - \frac{xdy}{yy}$ et $S = \frac{x}{y}$. Hinc oritur

$$x = Sy \quad \text{et} \quad U = n\sqrt{(1+SS)}$$

ideoque posito S constante erit

$$T = \int Udy = ny\sqrt{(1+SS)} \quad \text{et} \quad W = \left(\frac{dT}{dS}\right) = \frac{nyS}{\sqrt{(1+SS)}},$$

ita ut solutio nostrae quaestionis sit

$$py = \frac{nyS}{\sqrt{(1+SS)}} + f':S \quad \text{et} \quad z = ny\sqrt{(1+SS)} + f:S.$$

Cum igitur sit $S = \frac{x}{y}$, erit

$$z = n\sqrt{(xx+yy)} + f:\frac{x}{y},$$

ubi $f:\frac{x}{y}$ denotat functionem quamcunque nullius dimensionis ipsarum x et y.

EXEMPLUM 3

154. *Si posito $dz = pdx + qdy$ debeat esse $pxx + qyy = nxy$, functionis z indolem investigare.*

Cum sit $q = \frac{-pxx}{yy} + \frac{nx}{y}$, erit

$$V = \frac{-xx}{yy} \quad \text{et} \quad U = \frac{nx}{y}.$$

Quare ob $dS = M\left(dx - \frac{xxdy}{yy}\right)$ capiatur $M = \frac{1}{xx}$, ut fiat $S = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$.

Hinc erit

$$\frac{1}{x}=\frac{1}{y}-S \quad \text{et} \quad x=\frac{y}{1-Sy}$$

ideoque $U=\frac{n}{1-Sy}$. Sumto igitur S constante habebimus

$$T=\int\frac{ndy}{1-Sy}=-\frac{n}{S}l(1-Sy) \quad \text{et} \quad W=+\frac{n}{SS}l(1-Sy)+\frac{ny}{S(1-Sy)}.$$

Consequenter ob $S=\frac{x-y}{xy}$ et $1-Sy=\frac{y}{x}$ solutio praebet

$$z=\frac{-nxy}{x-y}l\frac{y}{x}+f:\frac{x-y}{xy}.$$

SCHOLION

155. Ex solutione huius problematis etiam haec quaestio latius patens resolvi potest. Sint P, Q, item V, U functiones quaecunque datae ipsarum x et y et quaeri oporteat functionem z, ut sit

$$dz=Pdx+Qdy+L(Vdx+Udy),$$

seu, quod eodem redit, functio L investigari debet, ut ista formula differentialis integrationem admittat.

Ad hoc praestandum quaeratur primo multiplicator M formulam $Vdx+Udy$ integrabilem efficiens ponaturque

$$dS=M(Vdx+Udy),$$

unde functio S reperietur per x et y expressa. Ex ea quaeratur valor ipsius x per y et S expressus, et cum sit

$$dz=Pdx+Qdy+\frac{LdS}{M},$$

hic ubique loco x valor ille substituatur; sit autem inde $dx=Edy+FdS$, unde etiam E et F innotescent, eritque

$$dz=EPdy+Qdy+FPdS+\frac{LdS}{M}.$$

Sumatur quantitas S pro constante sitque

$$T=\int(EP+Q)dy;$$

erit

$$z=T+f:S,$$

quod quidem ad solutionem sufficit. Sed ad L inveniendum differentietur haec expressio

$$dz = (EP + Q)dy + dS\left(\frac{dT}{dS}\right) + dSf' : S$$

ac necesse est fiat

$$FP + \frac{L}{M} = \left(\frac{dT}{dS}\right) + f' : S$$

ideoque

$$L = -FMP + M\left(\frac{dT}{dS}\right) + Mf' : S.$$

Ceterum ob permutabilitatem ipsarum p, x et q, y etiam hinc sequentia problemata resolvi possunt, quae propterea strictim percurram.

PROBLEMA 24

156. *Si posito $dz = pdx + qdy$ requiratur, ut sit $q = Vx + U$ existente tam V quam U functione quacunque data ipsarum p et y, investigare indolem functionis quaesitae z.*

SOLUTIO

Utamur formula

$$z = px + \int (qdy - xdp),$$

et cum loco q valore substituto sit

$$\int (qdy - xdp) = \int (Vxdy - xdp + Udy),$$

quam formulam integrabilem reddi oportet, sit ea brevitatis gratia $\mathfrak{h}$, et cum sit

$$d\mathfrak{h} = x(Vdy - dp) + Udy,$$

quaeratur primo multiplicator M formulam $Vdy - dp$ integrabilem reddens ponaturque

$$M(Vdy - dp) = dS$$

sicque S dabitur per y et p; unde p eliciatur per y et S expressum, quo valore ibi substituto erit

$$d\mathfrak{h} = \frac{xdS}{M} + Udy.$$

Iam sumto S constante sumatur integrale $\int U dy = T + f : S$ eritque

$$\frac{x}{M} = \left(\frac{dT}{dS}\right) + f' : S \quad \text{et} \quad \mathfrak{z} = T + f : S.$$

Solutio igitur per binas variabiles y et S ita se habebit

$$x = M\left(\frac{dT}{dS}\right) + Mf' : S \quad \text{et} \quad z = px + T + f : S,$$

ubi nunc quidem S per p et y datur.

PROBLEMA 25

157. *Si posito $dz = pdx + qdy$ requiratur, ut sit $p = Vy + U$ existentibus V et U functionibus datis ipsarum x et q, indolem functionis z investigare.*

SOLUTIO

Utamur iam forma

$$z = qy + \int (pdx - ydq)$$

ponaturque formula ad integrationem perducenda

$$\int (pdx - ydq) = \mathfrak{z}.$$

Hinc pro p valorem assumtum substituendo erit

$$d\mathfrak{z} = Vydx + Udx - ydq = y(Vdx - dq) + Udx.$$

Quaeramus multiplicatorem M, ut fiat

$$M(Vdx - dq) = dS,$$

ac tam M quam S erunt functiones ipsarum x et q, ex quarum posteriori valor ipsius q per x et S expressus eliciatur in sequenti operatione pro q substituendus. Scilicet cum nunc sit

$$d\mathfrak{z} = \frac{ydS}{M} + Udx,$$

sumto S constante quaeratur $T = \int U dx$ sitque

$$\mathfrak{z} = T + f : S,$$

unde colligitur

$$\frac{y}{M} = \left(\frac{dT}{dS}\right) + f' : S \quad \text{et} \quad z = qy + T + f : S,$$

ac nunc quidem pro S valorem in x et q restituere licet.

PROBLEMA 26

158. *Si posito $dz = pdx + qdy$ requiratur, ut sit $y = Vx + U$ existentibus V et U functionibus quibuscunque datis ipsarum p et q, indolem functionis z in genere investigare.*

SOLUTIO

Hic utendum est formula

$$z = px + qy - \int (xdp + ydq);$$

statuatur $\int (xdp + ydq) = \mathfrak{h}$ eritque pro y valorem praescriptum substituendo

$$d\mathfrak{h} = xdp + Vxdq + Udq.$$

Quaeratur iam multiplicator M formulam $dp + Vdq$ integrabilem reddens sitque

$$M(dp + Vdq) = dS,$$

ubi M et S per p et q dabuntur, et ex posteriori eliciatur valor ipsius p per q et S expressus, quo deinceps uti oportet. Scilicet cum sit

$$d\mathfrak{h} = \frac{xdS}{M} + Udq,$$

sumto S constante integretur formula Udq sitque $T = \int Udq$; erit $\mathfrak{h} = T + f : S$ hincque

$$\frac{x}{M} = \left(\frac{dT}{dS}\right) + f' : S \quad \text{et} \quad z = px + qy - T - f : S.$$

Omnia ergo per p et q, unde M, S et T cum $\left(\frac{dT}{dS}\right)$ dantur, ita determinabuntur, ut sit

$$x = M\left(\frac{dT}{dS}\right) + Mf' : S, \quad y = Vx + U \quad \text{et} \quad z = px + qy - T - f : S.$$

EXEMPLUM

159. *Si posito* $dz = pdx + qdy$ *debeat esse* $px + qy = apq$, *indolem functionis* z *investigare.*

Cum ergo sit $y = \frac{-px}{q} + ap$, erit

$$V = \frac{-p}{q} \quad \text{et} \quad U = ap.$$

Quia nunc esse debet $M\left(dp - \frac{pdq}{q}\right) = dS$, capiatur $M = \frac{1}{q}$ fitque $S = \frac{p}{q}$ et $p = Sq$. Hinc $U = aSq$ et sumto S constante

$$T = \int Udq = \frac{1}{2}aSqq$$

ideoque $\left(\frac{dT}{dS}\right) = \frac{1}{2}aqq$. Quocirca pro solutione habebimus

$$x = \frac{1}{2}aq + \frac{1}{q}f' : \frac{p}{q}, \quad y = \frac{1}{2}ap - \frac{p}{qq}f' : \frac{p}{q}$$

et

$$z = px + qy - \frac{1}{2}apq - f : \frac{p}{q} = \frac{1}{2}apq - f : \frac{p}{q}.$$

Per reductionem autem supra [§ 116] traditam habebimus

$$y = (aq - x)F' : \left(qx - \frac{1}{2}aqq\right) \quad \text{et} \quad z = qy + F : \left(qx - \frac{1}{2}aqq\right).$$

SCHOLION

160. Quatuor problemata haec coniunctim considerata admodum late patent atque pro formula $dz = pdx + qdy$ omnes relationes inter p, q, x et y complectuntur, in quibus vel x et y, vel p et y, vel x et q, vel p et q nusquam unam dimensionem superant. Ex quo saepe fieri potest, ut eadem quaestio per duo plurave horum quatuor problematum resolvi possit, veluti evenit in exemplo hoc postremo; in quo cum non solum x et y, sed etiam x et q, itemque p et y nusquam plus una dimensione occupent, id ad tria praecedentia problemata [§ 156—158] referri queat haecque conditio primo tantum problemati [§ 146] adversatur. Quodsi autem inter p, q, x, y haec

relatio praescribatur, ut esse debeat

$$\alpha px + \beta qy + ap + bq + mx + ny + c = 0,$$

resolutio per omnia quatuor problemata aeque institui potest. Verum etiam resolutiones inde ortae, etiamsi forma discrepent, tamen per reductionem ante expositam ad consensum revocari possunt.

At sequens casus latissime patens resolutionem quoque admittit, quem propterea evolvi conveniet.

PROBLEMA 27

161. *Si posito $dz = pdx + qdy$ inter p, q et x, y eiusmodi relatio detur, ut functio quaedam ipsarum p et x aequetur functioni cuipiam ipsarum q et y, functionis z indolem in genere investigare.*

SOLUTIO

Sit P functio illa ipsarum p et x et Q functio illa ipsarum q et y, quae inter se aequales esse debent. Cum igitur sit $P = Q$, ponatur utraque $= v$, ut sit $P = v$ et $Q = v$. Ex priori ergo p definire licebit per x et v, ex posteriori vero q per y et v; quo facto in formula $dz = pdx + qdy$ cum p sit functio ipsarum x et v, integretur pars pdx sumto v constante sitque $\int pdx = R$; simili modo cum q sit functio ipsarum y et v, integretur quoque altera pars qdy sumto v constante sitque $\int qdy = S$; erit ergo $R =$ functioni ipsarum x et v et $S =$ functioni ipsarum y et v. At sumto etiam v variabili sit

$$dR = pdx + Vdv \quad \text{et} \quad dS = qdy + Udv,$$

unde colligitur

$$dz = dR + dS - dv(V + U);$$

quae forma quia integrabilis esse debet, oportet sit $V + U = f' : v$. Quare solutio problematis his duabus aequationibus continebitur

$$V + U = f' : v \quad \text{et} \quad z = R + S - f : v.$$

Scilicet cum p, R et V dentur per x et v atque q, S et U per y et v, per aequationem priorem definitur v ex x et y, qui valor in altera substitutus determinabit functionem quaesitam z per x et y.

COROLLARIUM 1

162. Quoties ergo q eiusmodi functioni ipsarum p, x, y aequari debet, ut inde aequatio formari possit, ex cuius altera parte tantum binae litterae x et p, ex altera tantum binae reliquae y et q reperiantur, problema resolvi poterit.

COROLLARIUM 2

163. Si functio illa binarum litterarum p et x, quam posui P, ita sit comparata, ut posita ea $= v$ inde facilius x per p et v definiri possit, tum uti conveniet formula

$$z = px + \int (qdy - xdp)$$

et evolutio perinde se habebit atque ante.

COROLLARIUM 3

164. Simili modo si ex functione altera $Q = v$ quantitas y facilius per q et v definiatur, resolutio ex forma

$$z = qy + \int (pdx - ydq)$$

erit petenda. Sin autem utrumque eveniat, ut tam x per p et v quam y per q et v definiatur, utendum erit formula

$$z = px + qy - \int (xdp + ydq).$$

SCHOLION

165. Problema hoc innumerabiles complectitur casus in praecedentibus non comprehensos atque etiam eius solutio diverso nititur fundamento. Interim tamen longissime adhuc distamus a solutione problematis generalis, cui hoc caput est destinatum et quo in genere solutio desideratur, si inter quaternas quantitates p, q, x, y aequatio quaecunque proponatur, quae autem ob defectum Analyseos ne sperari quidem posse videtur. Contentos ergo nos esse oportet, si quam plurimos casus resolvere docuerimus. Quo autem vis huius problematis magis perspiciatur, aliquot exempla adiungamus.

EXEMPLUM 1

166. *Si posito $dz = pdx + qdy$ esse debeat $q = \frac{xxyy}{a^4 p}$, indolem functionis z investigare.*

Quia hic p, x et q, y separare licet, cum sit $\frac{aaq}{yy} = \frac{xx}{aap}$, ponatur $\frac{xx}{aap} = v = \frac{aaq}{yy}$, unde p per x et v et q per y et v ita definitur, ut sit

$$p = \frac{xx}{aav} \quad \text{et} \quad q = \frac{vyy}{aa}$$

ideoque

$$dz = \frac{xxdx}{aav} + \frac{vyydy}{aa}.$$

Hinc colligimus

$$z = \frac{x^3}{3aav} + \frac{vy^3}{3aa} + \frac{1}{3aa}\int\left(\frac{x^3 dv}{vv} - y^3 dv\right)$$

sicque $\frac{x^3}{vv} - y^3$ debet esse functio ipsius v. Ac posito

$$\frac{x^3}{vv} - y^3 = f' : v \quad \text{seu} \quad y^3 = \frac{x^3}{vv} - f' : v$$

erit

$$z = \frac{1}{3aa}\left(\frac{x^3}{v} + vy^3 + f : v\right).$$

COROLLARIUM

167. Hinc facillime v eliminatur, si ponatur $f' : v = \frac{b^3}{vv} - c^3$ hincque $f : v = \frac{-b^3}{v} - c^3 v$. Iam prior aequatio dat $y^3 - c^3 = \frac{x^3 - b^3}{vv}$, unde $vv = \frac{x^3 - b^3}{y^3 - c^3}$, et ob

$$3aaz = \frac{x^3 + vvy^3 - b^3 - c^3 vv}{v} = 2v(y^3 - c^3)$$

erit

$$z = \frac{2}{3aa}\sqrt{(x^3 - b^3)(y^3 - c^3)}.$$

EXEMPLUM 2

168. *Si posito $dz = pdx + qdy$ debeat esse $q = \frac{1}{b}\sqrt{(xx + yy - aapp)}$, investigare indolem functionis z.*

Conditio praescripta redit ad

$$bbqq - yy = xx - aapp = v,$$

unde elicimus

$$q = \frac{1}{b}\sqrt{(yy+v)} \quad \text{et} \quad p = \frac{1}{a}\sqrt{(xx-v)}.$$

Nunc vero est

$$\int p\,dx = \frac{1}{a}\int dx\sqrt{(xx-v)} = \frac{1}{2a}x\sqrt{(xx-v)} - \frac{v}{2a}\int\frac{dx}{\sqrt{(xx-v)}}$$

$$= \frac{x}{2a}\sqrt{(xx-v)} - \frac{v}{2a}l(x+\sqrt{(xx-v)}) = R;$$

simili modo est

$$\int q\,dy = \frac{y}{2b}\sqrt{(yy+v)} + \frac{v}{2b}l(y+\sqrt{(yy+v)}) = S.$$

Quare cum sit

$$V = \left(\frac{dR}{dv}\right) = \frac{-x}{4a\sqrt{(xx-v)}} - \frac{1}{2a}l(x+\sqrt{(xx-v)}) + \frac{v}{4a(x+\sqrt{(xx-v)})\sqrt{(xx-v)}},$$

quae reducitur ad

$$V = -\frac{1}{4a} - \frac{1}{2a}l(x+\sqrt{(xx-v)}),$$

similique modo

$$U = \left(\frac{dS}{dv}\right) = +\frac{1}{4b} + \frac{1}{2b}l(y+\sqrt{(yy+v)}),$$

ubi cum $V + U = f' : v$, erit

$$\frac{a-b}{4ab} + l\frac{(y+\sqrt{(yy+v)})^{\frac{1}{2b}}}{(x+\sqrt{(xx-v)})^{\frac{1}{2a}}} = f' : v,$$

unde valor ipsius v per x et y determinatur. Ex quo tandem colligitur

$$z = \frac{x}{2a}\sqrt{(xx-v)} + \frac{y}{2b}\sqrt{(yy+v)} + vl\frac{(y+\sqrt{(yy+v)})^{\frac{1}{2b}}}{(x+\sqrt{(xx-v)})^{\frac{1}{2a}}} - f : v$$

seu

$$z = \frac{x}{2a}\sqrt{(xx-v)} + \frac{y}{2b}\sqrt{(yy+v)} - \frac{(a-b)v}{4ab} + vf' : v - f : v.$$

SCHOLION

169. Haec solutio a formulis logarithmicis liberari potest hoc modo. Ponatur

$$f':v = lt + \frac{a-b}{4ab},$$

ut sit

$$t^{2ab} = \frac{(y+\sqrt{(yy+v)})^a}{(x+\sqrt{(xx-v)})^b},$$

unde v datur per t. Tum vero sit $v = tF':t$ et ob $dvf'':v = \frac{dt}{t}$ erit

$$\int vdvf'':v = vf':v - f:v = \int \frac{vdt}{t} = F:t$$

sicque erit

$$z = \frac{x}{2a}\sqrt{(xx-v)} + \frac{y}{2b}\sqrt{(yy+v)} - \frac{(a-b)v}{4ab} + F:t,$$

ubi est

$$v = tF':t \quad \text{et} \quad t^{2ab} = \frac{(y+\sqrt{(yy+v)})^a}{(x+\sqrt{(xx-v)})^b},$$

unde t et v per x et y definiri potest. Hinc statim patet, si capiatur $F':t = 0$, fore $v = 0$, $F:t = 0$ et $z = \frac{xx}{2a} + \frac{yy}{2b}$ hincque $p = \frac{x}{a}$ et $q = \frac{y}{b}$, quo pacto utique conditioni praescriptae satisfit.

Ceterum haec ratio quantitates logarithmicas elidendi maxime est notatu digna et in aliis casibus usum amplissimum habere potest.

EXEMPLUM 3

170. *Si posito $dz = pdx + qdy$ debeat esse $x^m y^n = Ap^\mu q^\nu$, indolem functionis z investigare.*

Statuatur ergo

$$\frac{x^m}{p^\mu} = \frac{Aq^\nu}{y^n} = v^{\mu\nu}$$

et hinc deducitur

$$p = \frac{x^{\frac{m}{\mu}}}{v^\nu} \quad \text{et} \quad q = \frac{1}{a} y^{\frac{n}{\nu}} v^\mu$$

posito $A = a^\nu$. Unde habebimus

$$\int p\,dx = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)v^{\nu}} + \frac{\mu\nu}{m+\mu}\int \frac{x^{\frac{m+\mu}{\mu}}\,dv}{v^{\nu+1}}$$

et

$$\int q\,dy = \frac{\nu y^{\frac{n+\nu}{\nu}} v^{\mu}}{(n+\nu)a} - \frac{\mu\nu}{(n+\nu)a}\int y^{\frac{n+\nu}{\nu}} v^{\mu-1}\,dv.$$

Quocirca erit

$$z = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)v^{\nu}} + \frac{\nu y^{\frac{n+\nu}{\nu}} v^{\mu}}{(n+\nu)a} + \frac{\mu\nu}{(m+\mu)(n+\nu)a}\int dv\left(\frac{(n+\nu)ax^{\frac{m+\mu}{\mu}}}{v^{\nu+1}} - (m+\mu)y^{\frac{n+\nu}{\nu}} v^{\mu-1}\right),$$

ita ut, si statuamus

$$\frac{x^{\frac{m+\mu}{\mu}}}{(m+\mu)v^{\nu+1}} - \frac{y^{\frac{n+\nu}{\nu}} v^{\mu-1}}{(n+\nu)a} = f':v,$$

futurum sit

$$z = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)v^{\nu}} + \frac{\nu y^{\frac{n+\nu}{\nu}} v^{\mu}}{(n+\nu)a} + \mu\nu f:v.$$

Pro casu simplicissimo ponamus $f':v = 0$ et $f:v = 0$ eritque

$$y^{\frac{n+\nu}{\nu}} v^{\mu+\nu} = \frac{(n+\nu)a}{m+\mu} x^{\frac{m+\mu}{\mu}} \quad \text{et} \quad v = \left(\frac{(n+\nu)ax^{\frac{m+\mu}{\mu}}}{(m+\mu)y^{\frac{n+\nu}{\nu}}}\right)^{\frac{1}{\mu+\nu}},$$

tum vero

$$z = \frac{1}{v^{\nu}}\left(\frac{\mu}{m+\mu}x^{\frac{m+\mu}{\mu}} + \frac{\nu}{(n+\nu)a}y^{\frac{n+\nu}{\nu}} v^{\mu+\nu}\right)$$

seu

$$z = \frac{\mu+\nu}{(m+\mu)v^{\nu}} x^{\frac{m+\mu}{\mu}} = (\mu+\nu)\left(\frac{x^{m+\mu}y^{n+\nu}}{(m+\mu)^{\mu}(n+\nu)^{\nu}A}\right)^{\frac{1}{\mu+\nu}}.$$

PROBLEMA 28

171. *Si posito* $dz = p\,dx + q\,dy$ *inter* p, q *et* x, y *eiusmodi detur relatio, ut* p *et* q *aequentur functionibus quibusdam ipsarum* x, y *et novae variabilis* v, *explorare casus, quibus indolem functionis* z *investigare licet.*

SOLUTIO

Cum sit p functio ipsarum x, y et v, spectatis y et v ut constantibus quaeratur integrale $\int p\,dx = P$ sitque sumtis omnibus variabilibus

$$dP = p\,dx + R\,dy + M\,dv,$$

unde, si pro pdx valor substituatur, erit

$$dz = dP + (q - R)dy - Mdv.$$

Quodsi iam eveniat, ut $q - R$ sit tantum functio ipsarum y et v exclusa x, sumta v constante quaeratur $\int (q - R)dy = T$ sitque deinceps

$$dT = (q - R)dy + Vdv.$$

Hinc valor ipsius $(q - R)dy$ ibi substitutus dabit

$$dz = dP + dT - (M + V)dv;$$

quae forma quia integrabilis esse debet, statuatur $M + V = f' : v$ eritque

$$z = P + T - f : v.$$

Ex operationibus autem susceptis dantur P, R, M per x, y et v, at T et V per y et v tantum; ac resolutio succedit, si modo in forma $q - R$ non amplius x continetur.

Pari ratione solutio succedet, si M tantum per y et v detur; tum enim ex y constante quaeratur $\int Mdv = L$ sitque $dL = Mdv + Ndy$; erit

$$dz = dP + (q - R + N)dy - dL$$

ponique conveniet $q - R + N = f' : y$, ut fiat

$$z = P - L + f : y.$$

Simili modo ab altera parte $\int qdy$ calculum incipere et prosequi licet.

Introducendo autem functionem ipsarum x, y et v indefinitam K negotium generalius confici poterit. Sit enim

$$dK = Fdx + Gdy + Hdv$$

ac consideretur haec forma

$$dz + dK = (p + F)dx + (q + G)dy + Hdv.$$

Nunc sumtis y et v constantibus quaeratur

$$\int (p + F)dx = P$$

sitque

$$dP = (p + F)dx + Rdy + Mdv,$$

unde habetur

$$dz + dK = dP + (q + G - R)dy + (H - M)dv.$$

Quodsi iam eveniat, ut vel $q + G - R$ vel $H - M$ tantum binas variabiles y et v exclusa x contineat, resolutio, ut ante est ostensum, absolvi poterit.

PROBLEMA 29

172. *Si posito $dz = pdx + qdy$ relatio detur inter binas formulas differentiales p, q et binas variabiles x et z vel y et z, solutionem problematis, quatenus fieri potest, perficere.*

SOLUTIO

Ponamus relationem dari inter p, q et x, z atque hunc casum facile ad praecedentem revocare licet. Consideretur enim haec formula

$$dy = \frac{dz - pdx}{q}$$

ex principali derivata voceturque

$$\frac{1}{q} = m \quad \text{et} \quad \frac{p}{q} = -n,$$

ut habeatur

$$dy = mdz + ndx,$$

et ob $q = \frac{1}{m}$ et $p = -\frac{n}{m}$ relatio proposita versabitur inter quaternas quantitates m, n, z et x ideoque quaestio omnino similis est earum, quas antea tractavimus, hoc tantum discrimine, quod hic quantitas y definiatur, cum ante esset z investigata. Quoniam autem ista determinatio per aequationes absolvitur, perinde est, utrum tandem inde z an y elicere velimus. Quodsi ergo hac reductione facta quaestio in casus ante pertractatos incidat, methodis quoque expositis resolvi poterit.

EXEMPLUM

173. *Si posito $dz = pdx + qdy$ debeat esse $qxz = aap$, indolem functionis z investigare.*

Consideretur formula $dy = \frac{dz}{q} - \frac{pdz}{q}$. Iam quia $\frac{p}{q} = \frac{xz}{aa}$, erit

$$dy = \frac{dz}{q} - \frac{xzdx}{aa} \quad \text{et} \quad y = \int\left(\frac{dz}{q} - \frac{xzdx}{aa}\right),$$

at est

$$\int\frac{xzdx}{aa} = \frac{xxz}{2aa} - \int\frac{xxdz}{2aa},$$

ergo

$$y = \int dz\left(\frac{1}{q} + \frac{xx}{2aa}\right) - \frac{xxz}{2aa}.$$

Ponatur ergo

$$\frac{1}{q} + \frac{xx}{2aa} = f':z;$$

erit

$$y = \frac{-xxz}{2aa} + f:z,$$

ex qua aequatione utique z per x et y definitur.

Si pro casu simpliciori sumamus $f:z = b + \alpha z$, erit

$$y - b = \left(\alpha - \frac{xx}{2aa}\right)z \quad \text{et} \quad z = \frac{2aa(y-b)}{2\alpha aa - xx};$$

et sumtis $\alpha = 0$ et $b = 0$ pro casu simplicissimo erit $z = \frac{-2aay}{xx}$. Hinc autem fit

$$p = \frac{+4aay}{x^3} \quad \text{et} \quad q = \frac{-2aa}{xx},$$

ergo

$$\frac{p}{q} = -\frac{2y}{x} \quad \text{et} \quad \frac{xz}{aa} = \frac{-2y}{x}.$$

CAPUT VI

DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER BINAS FORMULAS DIFFERENTIALES $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ ET OMNES TRES VARIABILES x, y, z QUAECUNQUE DATUR

PROBLEMA 30

174. *Si posito* $dz = pdx + qdy$ *debeat esse* $nz = px + qy$, *indolem functionis* z *in genere investigare.*

SOLUTIO

Ope relationis datae elidatur vel p vel q; scilicet cum sit $q = \frac{nz}{y} - \frac{px}{y}$, erit

$$dz = pdx + \frac{nzdy}{y} - \frac{pxdy}{y},$$

quae aequatio in hanc formam transfundatur

$$dz - \frac{nzdy}{y} = p\left(dx - \frac{xdy}{y}\right) = pyd.\frac{x}{y}.$$

Ut prius membrum $dz - \frac{nzdy}{y}$ integrabile reddatur, multiplicetur aequatio per $\frac{1}{z}$ funct. $\frac{z}{y^n}$ seu particulariter per $\frac{1}{y^n}$ eritque

$$d.\frac{z}{y^n} = py^{1-n}d.\frac{x}{y}.$$

Quo facto evidens est poni debere $py^{1-n} = f' : \frac{x}{y}$, ut fiat

$$\frac{z}{y^n} = f : \frac{x}{y} \quad \text{seu} \quad z = y^n f : \frac{x}{y}.$$

Unde patet fore z functionem homogeneam ipsarum x et y dimensionum numero existente $= n$.

Si in genere aequatio multiplicetur per $\frac{1}{z}$ funct. $\frac{z}{y^n}$, erit partis prioris integrale $F : \frac{z}{y^n}$, pro parte autem altera si ponatur $\frac{py}{z}$ funct. $\frac{z}{y^n} = f' : \frac{x}{y}$, erit

$$F : \frac{z}{y^n} = f : \frac{x}{y}$$

atque ut ante $\frac{z}{y^n}$ aequabitur functioni cuicunque ipsius $\frac{x}{y}$.

COROLLARIUM 1

175. Cum z aequetur functioni homogeneae n dimensionum ipsarum x et y, erunt p et q functiones $n-1$ dimensionum. Scilicet cum sit $z = y^n f : \frac{x}{y}$, erit

$$p = y^{n-1} f' : \frac{x}{y} \quad \text{et} \quad q = ny^{n-1} f : \frac{x}{y} - xy^{n-2} f' : \frac{x}{y},$$

unde fit manifesto $nz = px + qy$.

COROLLARIUM 2

176. Si p et q fuerint functiones $n-1$ dimensionum ipsarum x et y ac formula $pdx + qdy$ sit integrabilis seu $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$, tum integrale certo erit $\frac{px+qy}{n}$, quae proprietas nonnunquam insignem usum habere potest.

SCHOLION

177. Fundamentum huius solutionis in hoc consistit, quod aequatio integranda in duas partes resolvatur, quarum utraque ope certi multiplicatoris integrabilis reddi queat, unde deinceps una quantitas variabilis, cuius differentiale in aequatione non occurrit, determinetur. Hinc aequatio nostra

$$dz - \frac{nzdy}{y} = p\left(dx - \frac{xdy}{y}\right)$$

etiam ita repraesentari potest

$$\frac{dx}{y}-\frac{xdy}{yy}=\frac{1}{py}\left(dz-\frac{nzdy}{y}\right)=\frac{y^{n-1}}{p}\left(\frac{dz}{y^n}-\frac{nzdy}{y^{n+1}}\right)$$

seu

$$d.\frac{x}{y}=\frac{y^{n-1}}{p}\,d.\frac{z}{y^n}.$$

Sit ergo $\frac{y^{n-1}}{p}=F':\frac{z}{y^n}$ eritque $\frac{x}{y}=F:\frac{z}{y^n}$ ac vicissim $\frac{z}{y^n}=f:\frac{x}{y}$ ut ante.

Possumus etiam statim z ex calculo elidere; cum enim sit $nz=px+qy$, erit

$$ndz=pdx+qdy+xdp+ydq.$$

At est

$$ndz=npdx+nqdy$$

per hypothesin ideoque

$$(n-1)pdx-xdp+(n-1)qdy-ydq=0$$

seu

$$x^n\left(\frac{(n-1)pdx}{x^n}-\frac{dp}{x^{n-1}}\right)+y^n\left(\frac{(n-1)qdy}{y^n}-\frac{dq}{y^{n-1}}\right)=0,$$

quae reducitur ad hanc formam

$$-x^n d.\frac{p}{x^{n-1}}-y^n d.\frac{q}{y^{n-1}}=0 \quad \text{seu} \quad d.\frac{q}{y^{n-1}}=-\frac{x^n}{y^n}\,d.\frac{p}{x^{n-1}}.$$

Statuatur

$$\frac{x^n}{y^n}=-f':\frac{p}{x^{n-1}};$$

erit

$$\frac{q}{y^{n-1}}=f:\frac{p}{x^{n-1}}.$$

Vel posito $\frac{x}{y}=v$ si ob $v^n=-f':\frac{p}{x^{n-1}}$ reciproce ponatur

$$\frac{p}{x^{n-1}}=u=\frac{1}{v^{n-1}}F':v,$$

ut sit $f':u=-v_n$, reperietur

$$\int duf':u=f:u=nF:v-vF':v.$$

Hinc

$$p = \frac{x^{n-1}}{v^{n-1}} F' : v = y^{n-1} F' : \frac{x}{y}$$

et

$$q = y^{n-1} f : u = n y^{n-1} F : \frac{x}{y} - x y^{n-2} F' : \frac{x}{y}$$

ideoque

$$nz = px + qy = ny^n F : \frac{x}{y} \quad \text{seu} \quad z = y^n F : \frac{x}{y}$$

ut ante.

PROBLEMA 31

178. *Si posito $dz = pdx + qdy$ debeat esse $\alpha px + \beta qy = nz$, indolem functionis z investigare.*

SOLUTIO

Ex conditione praescripta eliciatur ut ante $q = \frac{nz}{\beta y} - \frac{\alpha px}{\beta y}$ eritque

$$dz - \frac{nz dy}{\beta y} = p dx - \frac{\alpha p x dy}{\beta y},$$

quae aequatio per $y^{\frac{n}{\beta}}$ divisa dat

$$d. \frac{z}{y^{n:\beta}} = \frac{p}{y^{n:\beta}} \left(dx - \frac{\alpha x dy}{\beta y} \right) = \frac{p y^{\alpha:\beta}}{y^{n:\beta}} d. \frac{x}{y^{\alpha:\beta}}.$$

Quodsi ergo ponamus

$$p y^{(\alpha - n):\beta} = f' : \frac{x}{y^{\alpha:\beta}},$$

habebimus solutionem

$$z = y^{n:\beta} f : \frac{x}{y^{\alpha:\beta}}.$$

At functio ipsius $\frac{x}{y^{\alpha:\beta}}$ reducitur ad functionem ipsius $\frac{x^\beta}{y^\alpha}$, unde z etiam ita per x et y determinatur, ut sit

$$z = y^{n:\beta} f : \frac{x^\beta}{y^\alpha} \quad \text{vel etiam} \quad z^{1:n} = y^{1:\beta} f : \frac{x^{1:\alpha}}{y^{1:\beta}}.$$

Quodsi ergo quantitates $x^{1:\alpha}$ et $y^{1:\beta}$ unam dimensionem constituere censeantur,

$z^{1:n}$ aequabitur earundem functioni unius dimensionis, ipsa autem quantitas z earundem functioni n dimensionum. Vel sumta pro z functione quacunque homogenea n dimensionum binarum variabilium t et u scribatur deinde $t = x^{1:\alpha}$ et $u = y^{1:\beta}$ ac prodibit functio conveniens pro z.

PROBLEMA 32

179. *Si posito $dz = pdx + qdy$ debeat esse $Z = pX + qY$ denotante Z functionem ipsius z, X ipsius x et Y ipsius y, indolem functionis z in genere investigare.*

SOLUTIO

Ex conditione praescripta elicitur $q = \frac{Z}{Y} - \frac{pX}{Y}$, qui valor substitutus praebet

$$dz - \frac{Zdy}{Y} = p\left(dx - \frac{Xdy}{Y}\right)$$

hincque

$$\frac{dz}{Z} - \frac{dy}{Y} = \frac{p}{Z}\left(dx - \frac{Xdy}{Y}\right) = \frac{pX}{Z}\left(\frac{dx}{X} - \frac{dy}{Y}\right),$$

ubi iam resolutio est manifesta. Statuatur scilicet

$$\frac{pX}{Z} = f':\left(\int\frac{dx}{X} - \int\frac{dy}{Y}\right)$$

eritque

$$\int\frac{dz}{Z} - \int\frac{dy}{Y} = f:\left(\int\frac{dx}{X} - \int\frac{dy}{Y}\right),$$

unde valor ipsius z per x et y definitur.

COROLLARIUM 1

180. Hic ergo z ita per x et y definiri debet, ut, si X, Y et Z datae sint functiones singillatim ipsarum x, y et z, fiat

$$X\left(\frac{dz}{dx}\right) + Y\left(\frac{dz}{dy}\right) = Z,$$

cuius ergo aequationis resolutionem hic invenimus hac aequatione finita contentam

$$\int\frac{dz}{Z}=\int\frac{dy}{Y}+f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right).$$

COROLLARIUM 2

181. Quemadmodum autem hic valor conditioni problematis satisfaciat, ex eius differentiatione statim patet. Cum enim sit

$$\frac{dz}{Z}=\frac{dy}{Y}+\left(\frac{dx}{X}-\frac{dy}{Y}\right)f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right),$$

erit

$$\left(\frac{dz}{dx}\right)=\frac{Z}{X}f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right) \quad \text{et} \quad \left(\frac{dz}{dy}\right)=\frac{Z}{Y}-\frac{Z}{Y}f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right),$$

unde fit

$$X\left(\frac{dz}{dx}\right)+Y\left(\frac{dz}{dy}\right)=Z.$$

SCHOLION

182. Solutio ergo eodem modo, ut fecimus, sine introductione novarum litterarum p et q absolvi potest retinendo earum loco valores differentiales $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$; facilius autem singulae litterae scribuntur calculusque fit brevior.

Ceterum ex hoc problematum genere, ubi omnes tres variabiles x, y et z praeter binos valores differentiales p et q in determinationem ingrediuntur, paucissima resolvere licet; ac praeter hoc, quod tractavimus, vix unum aut alterum insuper adiungere poterimus. Unde hic insignia adhuc calculi incrementa desiderantur. Quo autem huius problematis vis penitius inspiciatur, nonnulla exempla subiungamus.

EXEMPLUM 1

183. *Si posito $dz=pdx+qdy$ debeat esse $zz=pxx+qyy$, indolem functionis z in genere investigare.*

Hic ergo est $Z=zz$, $X=xx$ et $Y=yy$, unde habemus

$$\int\frac{dx}{X}=-\frac{1}{x}, \quad \int\frac{dy}{Y}=-\frac{1}{y} \quad \text{et} \quad \int\frac{dz}{Z}=-\frac{1}{z},$$

quibus valoribus substitutis pro solutione adipiscimur

$$-\frac{1}{z} = -\frac{1}{y} + f:\left(\frac{1}{y} - \frac{1}{x}\right) \quad \text{seu} \quad z = \frac{y}{1 - yf:\left(\frac{1}{y} - \frac{1}{x}\right)}.$$

Sumatur ergo functio quaecunque quantitatis $\frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$; quae si ponatur V, erit

$$z = \frac{y}{1 - Vy}.$$

Veluti si ponamus $V = \frac{n}{y} - \frac{n}{x}$, erit $\frac{1}{z} = \frac{1}{y} - \frac{n}{y} + \frac{n}{x} = \frac{ny - (n-1)x}{xy}$ hincque

$$z = \frac{xy}{ny - (n-1)x},$$

unde fit

$$p = \left(\frac{dz}{dx}\right) = \frac{nyy}{(ny - (n-1)x)^2} \quad \text{et} \quad q = \left(\frac{dz}{dy}\right) = \frac{-(n-1)xx}{(ny - (n-1)x)^2}$$

sicque

$$pxx + qyy = \frac{xxyy}{(ny - (n-1)x)^2} = zz.$$

EXEMPLUM 2

184. *Si posito* $dz = pdx + qdy$ *debeat esse* $\frac{n}{z} = \frac{p}{x} + \frac{q}{y}$, *indolem functionis* z *investigare.*

Cum hic sit $X = \frac{1}{x}$, $Y = \frac{1}{y}$ et $Z = \frac{n}{z}$, erit

$$\int \frac{dx}{X} = \frac{1}{2}xx, \quad \int \frac{dy}{Y} = \frac{1}{2}yy \quad \text{et} \quad \int \frac{dz}{Z} = \frac{1}{2n}zz,$$

unde solutio ita erit comparata

$$\frac{1}{2n}zz = \frac{1}{2}yy + f:(xx - yy) \quad \text{sive} \quad zz = nyy + f:(xx - yy);$$

non enim est necesse functionem per $2n$ multiplicari, cum ea omnes operationes iam per se involvat.

Si pro hac functione sumatur $\alpha(xx - yy)$, habebitur solutio particularis

$$zz = \alpha xx + (n - \alpha)yy \quad \text{et} \quad z = \sqrt{(\alpha xx + (n - \alpha)yy)}$$

hincque

$$p=\left(\frac{dz}{dx}\right)=\frac{\alpha x}{\sqrt{(\alpha xx+(n-\alpha)yy)}} \quad \text{et} \quad q=\left(\frac{dz}{dy}\right)=\frac{(n-\alpha)y}{\sqrt{(\alpha xx+(n-\alpha)yy)}}$$

seu $\frac{p}{x}=\frac{\alpha}{z}$ et $\frac{q}{y}=\frac{n-\alpha}{z}$ ideoque

$$\frac{p}{x}+\frac{q}{y}=\frac{n}{z}.$$

PROBLEMA 32 [a][1]

185. *Si posito* $dz=pdx+qdy$ *debeat esse* $q=pT+V$ *existente* T *functione quacunque ipsarum* x *et* y *ac* V *functione ipsarum* y *et* z, *investigare indolem functionis* z.

SOLUTIO

Substituto loco q valore praescripto huic aequationi inducatur forma

$$dz-Vdy=p(dx+Tdy).$$

Cum iam V tantum binas variabiles y et z involvat, dabitur multiplicator M prius membrum $dz-Vdy$ integrabile reddens; ponatur ergo

$$M(dz-Vdy)=dS.$$

Simili modo quia T tantum x et y continet, dabitur multiplicator L membrum quoque posterius $dx+Tdy$ integrabile efficiens; sit igitur

$$L(dx+Tdy)=dR,$$

ita ut nunc sint R et S functiones cognitae, illa ipsarum x et y, haec vero ipsarum y et z. Hinc nostra aequatio induet hanc formam

$$\frac{dS}{M}=\frac{pdR}{L} \quad \text{seu} \quad dS=\frac{pMdR}{L},$$

1) In editione principe falso numerus 32 iteratur. F. E.

cuius integrabilitas necessario postulat, ut sit $\frac{pM}{L}$ functio ipsius R. Ponamus ergo

$$\frac{pM}{L} = f' : R$$

eritque

$$S = f : R,$$

qua aequatione relatio inter z et x, y definitur.

COROLLARIUM 1

186. In hoc problemate praecedens tanquam casus particularis continetur; cum enim ibi esset $Z = pX + qY$, erit $q = -\frac{X}{Y}p + \frac{Z}{Y}$ ideoque huius problematis applicatione facta fit $T = -\frac{X}{Y}$ et $V = \frac{Z}{Y}$.

COROLLARIUM 2

187. Quanquam autem hoc problema infinite latius patet quam praecedens, arctissimis tamen adhuc limitibus continetur neque eius ope vel hunc casum simplicissimum $z = py + qx$ resolvere licet.

SCHOLION

188. Omnino est haec forma $z = py + qx$ digna notatu, quod nulla ratione hactenus cognita resolvi posse videtur. Sive enim inde eliciatur $q = \frac{z - py}{x}$, unde fit

$$dz - \frac{z\,dy}{x} = p\left(dx - \frac{y\,dy}{x}\right),$$

sive simili modo p, nulla via ad solutionem patet; cuius difficultatis causa in hoc manifesto est posita, quod formula $dz - \frac{z\,dy}{x}$ nullo multiplicatore integrabilis reddi potest seu quod haec aequatio $dz - \frac{z\,dy}{x} = 0$ plane est impossibilis, cum x perinde sit variabilis atque y et z. Supra scilicet [§ 6] iam notavi non omnes aequationes differentiales inter ternas variabiles esse possibiles simulque characterem possibilitatis exhibui, qui pro tali forma

$$dz + Pdx + Qdy = 0$$

huc reducitur, ut sit

$$P\left(\frac{dQ}{dz}\right)-Q\left(\frac{dP}{dz}\right)=\left(\frac{dQ}{dx}\right)-\left(\frac{dP}{dy}\right).$$

Nostro iam casu est $P=0$ et $Q=\frac{-z}{x}$, unde hic character dat $0=\frac{z}{xx}$; quod cum sit falsum, etiam aequatio illa $dz-\frac{zdy}{x}=0$ est impossibilis, quod quidem per se est manifestum.

Verum tamen pro hoc casu $z=py+qx$ solutio particularis est obvia, scilicet $z=n(x+y)$, unde fit $p=q=n$. Deinceps autem [§ 195] methodum dabimus ex huiusmodi solutione particulari generalem eruendi.

EXEMPLUM 1

189. *Si posito $dz=pdx+qdy$ debeat esse $py+qx=\frac{nxz}{y}$, indolem functionis z investigare.*

Cum hinc sit $q=-\frac{py}{x}+\frac{nz}{y}$, erit

$$T=\frac{-y}{x} \quad \text{et} \quad V=\frac{nz}{y},$$

unde fit

$$dS=M\left(dz-\frac{nzdy}{y}\right) \quad \text{et} \quad dR=L\left(dx-\frac{ydy}{x}\right).$$

Sumatur ergo $M=\frac{1}{y^n}$, ut fiat $S=\frac{z}{y^n}$, et $L=2x$, ut fiat $R=xx-yy$. Quocirca hanc adipiscimur solutionem

$$\frac{z}{y^n}=f:(xx-yy) \quad \text{seu} \quad z=y^n f:(xx-yy).$$

EXEMPLUM 2

190. *Si posito $dz=pdx+qdy$ debeat esse $pxx+qyy=nyz$, definire indolem functionis z.*

Cum ergo sit $q=-\frac{pxx}{yy}+\frac{nz}{y}$, erit

$$T=\frac{-xx}{yy} \quad \text{et} \quad V=\frac{nz}{y}$$

sicque hic casus in nostro problemate continetur. Unde colligi oportet

$$dR = L\left(dx - \frac{xxdy}{yy}\right) \quad \text{et} \quad dS = M\left(dz - \frac{nzdy}{y}\right).$$

Quare sumto $L = \frac{1}{xx}$ fit $R = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$ et sumto $M = \frac{1}{y^n}$ fit $S = \frac{z}{y^n}$ ideoque solutio prodit ista

$$\frac{z}{y^n} = f : \frac{x-y}{xy} \quad \text{et} \quad z = y^n f : \frac{x-y}{xy}.$$

PROBLEMA 33

191. *Si posito $dz = pdx + qdy$ debeat esse $p = qT + V$ existente T functione ipsarum x et y, at V functione ipsarum x et z, indolem functionis z investigare.*

SOLUTIO

Simili modo ut ante si loco p valor praescriptus substituatur, obtinebitur

$$dz - Vdx = q(dy + Tdx).$$

Iam ob indolem functionum V et T sequentes integrationes instituere licebit

$$M(dz - Vdx) = dS \quad \text{et} \quad N(dy + Tdx) = dR,$$

unde fit

$$\frac{dS}{M} = \frac{qdR}{N} \quad \text{seu} \quad dS = \frac{Mq}{N} dR.$$

Atque hinc facillime colligitur haec solutio

$$\frac{Mq}{N} = f' : R \quad \text{et} \quad S = f : R.$$

PROBLEMA 34

192. *Si posito $dz = pdx + qdy$ debeat esse $z = Mp + Nq$ existentibus M et N functionibus quibusvis binarum variabilium x et y, ex quadam solutione particulari, quam constat esse $z = V$, indolem functionis z in genere determinare.*

SOLUTIO

Valor iste particularis V, qui est functio ipsarum x et y, differentietur sitque

$$dV = Pdx + Qdy;$$

qui valor quia loco z substitutus satisfacit, ubi fit $p = P$ et $q = Q$, erit per hypothesin

$$V = MP + NQ.$$

Iam generatim ponatur $z = Vf : T$ sitque

$$dT = Rdx + Sdy$$

et nunc quaeri oportet hanc functionem T. Ex differentiatione autem eruimus

$$p = \left(\frac{dz}{dx}\right) = Pf : T + VRf' : T \quad \text{et} \quad q = \left(\frac{dz}{dy}\right) = Qf : T + VSf' : T.$$

Quare cum sit $z = Mp + Nq = Vf : T$, erit

$$Vf : T = (MP + NQ)f : T + V(MR + NS)f' : T$$

et ob $V = MP + NQ$ per hypothesin habebitur $MR + NS = 0$, hinc

$$dT = R\left(dx - \frac{Mdy}{N}\right).$$

Iam nosse non oportet R, sed sufficit considerari formulam $Ndx - Mdy$, quae ope multiplicatoris cuiusdam integrabilis reddi potest. Solutio ergo facillime huc redit, ut ex conditione praescripta $z = Mp + Nq$ formetur aequatio realis

$$dT = R(Ndx - Mdy);$$

invento enim multiplicatore idoneo R per integrationem reperitur quantitas T, qua inventa erit $z = Vf : T$.

ALITER

Facilius valor generalis hoc modo invenitur. Ob valorem ipsius z cognitum V statuatur $z = Vv$ sitque $dv = rdx + sdy$; erit

$$p = Pv + Vr \quad \text{et} \quad q = Qv + Vs$$

ideoque

$$z = Mp + Nq = (MP + NQ)v + V(Mr + Ns) = Vv.$$

At est $V = MP + NQ$, ergo

$$Mr + Ns = 0 \quad \text{seu} \quad s = -\frac{Mr}{N},$$

unde fit

$$dv = r\left(dx - \frac{Mdy}{N}\right) = \frac{r}{N}(Ndx - Mdy)$$

Statuatur ergo idoneum multiplicatorem investigando

$$R(Ndx - Mdy) = dT;$$

erit $dv = \frac{r}{NR}dT$, ex quo colligitur

$$\frac{r}{NR} = f' : T \quad \text{et} \quad v = f : T,$$

ita ut in genere sit ut ante $z = Vv$.

COROLLARIUM 1

193. Proposita ergo conditione $z = Mp + Nq$ ut sit $dz = pdx + qdy$, statim consideretur aequatio differentialis $R(Ndx - Mdy) = dT$, unde tam multiplicator R quam inde integrale T reperitur; haecque operatio non pendet a valore particulari cognito V.

COROLLARIUM 2

194. Inventa autem quantitate T si undecunque innotuerit solutio particulariter satisfaciens $z = V$, erit solutio generalis $z = Vf : T$. Probe autem notetur ex solutione particulari generalem elici non posse, nisi conditio praescripta sit huiusmodi $z = Mp + Nq$.

EXEMPLUM 1

195. *Si posito $dz = pdx + qdy$ debeat esse $z = py + qx$, ex valore particulari $z = x + y$ generalem definire.*

Cum hic sit $M = y$ et $N = x$, habebimus hanc aequationem

$$R(xdx - ydy) = dT$$

hincque

$$T = f:(xx - yy);$$

ergo solutio generalis erit

$$z = (x + y)f:(xx - yy).$$

EXEMPLUM 2

196. *Si posito $dz = pdx + qdy$ debeat esse $z = p(x + y) + q(y - x)$, ex valore particulari $z = \sqrt{(xx + yy)}$ generalem invenire.*

Ob $M = x + y$ et $N = y - x$ formula $Ndx - Mdy$ deducit ad hanc aequationem

$$R(ydx - xdx - xdy - ydy) = dT.$$

Sumatur $R = \frac{1}{xx + yy}$, ut sit

$$dT = \frac{ydx - xdy}{xx + yy} - \frac{xdx + ydy}{xx + yy};$$

erit

$$T = \text{Ang. tang.} \frac{x}{y} - \frac{1}{2} l(xx + yy).$$

Atque ex valore hoc dupliciter transcendente erit

$$z = \sqrt{(xx + yy)} f:T$$

simulque patet nullum alium dari valorem particularem, qui sit algebraicus, praeter datum $z = \sqrt{(xx + yy)}$.

EXEMPLUM 3

197. *Si posito $dz = pdx + qdy$ debeat esse $z = p(\alpha x + \beta y) + q(\gamma x + \delta y)$, ex invento valore particulari $z = V$ indolem functionis z in genere definire.*

Hic est $M = \alpha x + \beta y$ et $N = \gamma x + \delta y$, unde deducimur ad hanc aequationem

$$R((\gamma x + \delta y)dx - (\alpha x + \beta y)dy) = dT,$$

ubi ob formam homogeneam debet esse

$$R = \frac{1}{\gamma xx + (\delta - \alpha)xy - \beta yy},$$

ut sit

$$dT = \frac{(\gamma x + \delta y)dx - (\alpha x + \beta y)dy}{\gamma xx + (\delta - \alpha)xy - \beta yy},$$

ad quod integrale inveniendum ponatur $y = ux$ ac prodibit

$$dT = \frac{dx}{x} - \frac{(\alpha + \beta u)du}{\gamma + (\delta - \alpha)u - \beta uu}.$$

Sit

$$\int \frac{(\alpha + \beta u)du}{\gamma + (\delta - \alpha)u - \beta uu} = lU;$$

erit $T = lx - lU$, et cum functio ipsius T sit etiam functio ipsius $\frac{x}{U}$, erit in genere $z = Vf:\frac{x}{U}$. Patet autem, cum U sit functio ipsius $u = \frac{y}{x}$, fore U functionem homogeneam nullius dimensionis ipsarum x et y ideoque $\frac{x}{U}$ functionem unius dimensionis.

SCHOLION

198. Hoc ergo exemplo difficultas restat, quomodo solutio particularis $z = V$ obtineri queat; nisi enim una saltem huiusmodi solutio particularis constet, solutio generalis ne absolvi quidem potest. Pro hoc autem casu solutionem particularem sequenti modo elicere licet; qui cum aliquid singulare habeat, nullum est dubium, quin eius ope hoc calculi genus haud parum adiumenti sit consecuturum.

PROBLEMA 35

199. *Si posito $dz = pdx + qdy$ debeat esse $z = p(\alpha x + \beta y) + q(\gamma x + \delta y)$, valorem particularem investigare, qui loco z substitutus huic conditioni satisfaciat.*

SOLUTIO

Negotium hoc succedet, si pro z eiusmodi valorem quaeramus, qui sit functio nullius dimensionis ipsarum x et y, seu posito $y = ux$, qui sit functio ipsius u tantum. Ponamus ergo $z = f:u = f:\frac{y}{x}$ eritque $f':u = \frac{dz}{du}$; at ob $du = \frac{dy}{x} - \frac{ydx}{xx}$ erit

$$dz = \left(\frac{dy}{x} - \frac{udx}{x}\right) f':u,$$

hinc

$$p = -\frac{u}{x} f' : u = -\frac{udz}{xdu} \quad \text{et} \quad q = \frac{1}{x} f' : u = \frac{dz}{xdu}.$$

Quibus valoribus pro p et q substitutis conditio praescripta praebet

$$z = x(\alpha + \beta u)p + x(\gamma + \delta u)q = \frac{-udz(\alpha + \beta u) + dz(\gamma + \delta u)}{du},$$

unde fit

$$\frac{dz}{z} = \frac{du}{\gamma + (\delta - \alpha)u - \beta uu}.$$

Ponamus

$$\int \frac{du}{\gamma + (\delta - \alpha)u - \beta uu} = lV,$$

ut fiat $z = V$, eritque V valor particularis pro z satisfaciens.

COROLLARIUM 1

200. Invento hoc valore V praecedentis exempli ope solutio generalis facile invenitur. Erit scilicet $z = Vf : \frac{x}{U}$ existente

$$\frac{dU}{U} = \frac{(\alpha + \beta u)du}{\gamma + (\delta - \alpha)u - \beta uu},$$

unde patet quantitatem U ex ipso valore particulari V inveniri posse.

COROLLARIUM 2

201. Erit enim

$$lU = -l\sqrt{(\gamma + (\delta - \alpha)u - \beta uu)} + \int \frac{\frac{1}{2}(\delta + \alpha)du}{\gamma + (\delta - \alpha)u - \beta uu}$$

ideoque

$$lU = -l\sqrt{(\gamma + (\delta - \alpha)u - \beta uu)} + \frac{1}{2}(\alpha + \delta)lV$$

sive

$$U = \frac{V^{\frac{1}{2}(\alpha + \delta)}}{\sqrt{(\gamma + (\delta - \alpha)u - \beta uu)}},$$

hinc

$$\frac{x}{U} = \frac{\sqrt{(\gamma xx + (\delta - \alpha)xy - \beta yy)}}{V^{\frac{1}{2}(\alpha + \delta)}}.$$

COROLLARIUM 3

202. Quocirca invento valore particulari $z = V$, ut sit

$$\frac{dV}{V} = \frac{du}{\gamma + (\delta - \alpha)u - \beta uu}$$

existente $u = \frac{y}{x}$, erit valor generaliter satisfaciens

$$z = Vf : \frac{\gamma xx + (\delta - \alpha)xy - \beta yy}{V^{\alpha+\delta}} = Vf : \frac{x(\gamma x + \delta y) - y(\alpha x + \beta y)}{V^{\alpha+\delta}}.$$

COROLLARIUM 4

203. Hinc colligitur alius valor particularis, qui semper est algebraicus; erit is scilicet

$$z = (x(\gamma x + \delta y) - y(\alpha x + \beta y))^{\frac{1}{\alpha+\delta}}$$

vel eius multiplum quodcunque. Nisi autem V sit quantitas algebraica, omnes reliqui valores erunt transcendentes et in hac forma contenti

$$z = (x(\gamma x + \delta y) - y(\alpha x + \beta y))^{\frac{1}{\alpha+\delta}} f : \frac{x(\gamma x + \delta y) - y(\alpha x + \beta y)}{V^{\alpha+\delta}}.$$

SCHOLION

204. Unicus casus, quo $\delta = -\alpha$ et conditio proposita

$$z = p(\alpha x + \beta y) + q(\gamma x - \alpha y),$$

peculiarem evolutionem postulat. Primo autem posito $u = \frac{y}{x}$ pro valore particulari $z = V$ erit

$$lV = \int \frac{du}{\gamma - 2\alpha u - \beta uu}.$$

Tum vero ob $\frac{dU}{U} = \frac{(\alpha + \beta u)du}{\gamma - 2\alpha u - \beta uu}$ erit

$$U = \frac{1}{\sqrt{(\gamma - 2\alpha u - \beta uu)}} \quad \text{et} \quad \frac{x}{U} = \sqrt{(\gamma xx - 2\alpha xy - \beta yy)},$$

ita ut iam valor generalis sit

$$z = Vf:(\gamma xx - 2\alpha xy - \beta yy).$$

Per se enim manifestum est formam $f:\sqrt{T}$ exprimi posse per $f:T$. Nisi ergo V sit functio algebraica, hoc casu nulla solutio particularis algebraica locum habet.

EXEMPLUM 1

205. *Si posito $dz = pdx + qdy$ debeat esse $nz = py - qx$, indolem functionis z investigare.*

Comparatione cum forma nostra generali instituta fit

$$\alpha = 0, \quad \beta = \frac{1}{n}, \quad \gamma = -\frac{1}{n}, \quad \delta = 0.$$

Hic ergo casus ob $\delta = -\alpha$ pertinet ad paragraphum praecedentem, unde fit

$$lV = \int \frac{ndu}{-1-uu} = -n \text{ Ang. tang. } u.$$

Cum igitur sit $u = \frac{y}{x}$, forma generalis est

$$z = e^{-n \text{ Ang. tang.} \frac{y}{x}} f:(xx + yy).$$

EXEMPLUM 2

206. *Si posito $dz = pdx + qdy$ debeat esse $z = p(x + y) - q(x + y)$, indolem functionis z investigare.*

Comparatione facta fit

$$\alpha = 1, \quad \beta = 1, \quad \gamma = -1, \quad \delta = -1$$

hincque

$$lV = \int \frac{du}{-1-2u-uu} = \frac{1}{1+u} \quad \text{et} \quad V = e^{\frac{1}{1+u}}$$

et solutio generalis est

$$z = e^{\frac{x}{x+y}} f:(x + y).$$

EXEMPLUM 3

207. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = p(x - 2y) + q(2x - 3y)$, *indolem functionis* z *investigare.*

Cum ergo hic sit

$$\alpha = 1, \quad \beta = -2, \quad \gamma = 2 \quad \text{et} \quad \delta = -3,$$

erit primo

$$lV = \int \frac{du}{2 - 4u + 2uu} = \frac{1}{2(1-u)} = \frac{x}{2(x-y)},$$

et quia non est $\delta = -\alpha$, solutio generalis statim prodit

$$z = (2xx - 4xy + 2yy)^{-\frac{1}{2}} f\colon \frac{2xx - 4xy + 2yy}{V^{-2}}$$

et ob $V = e^{\frac{x}{2(x-y)}}$ erit

$$z = \frac{1}{x-y} f\colon (x-y)^2 e^{\frac{x}{x-y}}.$$

Unde solutio simplicissima est $z = \frac{1}{x-y}$.

SCHOLION

208. Hic merito quaerimus, quo pacto haec solutio generalis statim sine adiumento solutionis specialis inveniri potuisset; sequenti autem modo ista investigatio instituenda videtur.

Cum sit

$$p(\alpha x + \beta y) = z - q(\gamma x + \delta y) \quad \text{et} \quad q(\gamma x + \delta y) = z - p(\alpha x + \beta y),$$

si uterque valor seorsim in forma

$$dz = pdx + qdy$$

substituatur, prodibunt binae sequentes aequationes

$$(\alpha x + \beta y)dz = zdx - q(\gamma x + \delta y)dx + q(\alpha x + \beta y)dy,$$
$$(\gamma x + \delta y)dz = zdy + p(\gamma x + \delta y)dx - p(\alpha x + \beta y)dy.$$

Multiplicetur prior indefinite per M, posterior per N, et productorum summa

dabit

$$dz(M(\alpha x + \beta y) + N(\gamma x + \delta y)) - z(Mdx + Ndy)$$
$$= (Np - Mq)((\gamma x + \delta y)dx - (\alpha x + \beta y)dy),$$

ubi iam M et N ita capi debent, ut prius membrum integrationem admittat; tum enim eius integrale aequabitur functioni cuicunque quantitatis

$$\int \frac{(\gamma x + \delta y)dx - (\alpha x + \beta y)dy}{\gamma xx + (\delta - \alpha)xy - \beta yy},$$

quam supra (§ 197) definire docuimus; unde patet integrale fieri $= f : \frac{x}{U}$. Manifestum autem est M et N eiusmodi functiones esse oportere, ut haec aequatio fiat possibilis

$$\frac{dz}{z} = \frac{Mdx + Ndy}{M(\alpha x + \beta y) + N(\gamma x + \delta y)}$$

seu ut membrum posterius integrationem admittat; quodsi enim eius integrale sit $= lV$, erit $\frac{z}{V} = f : \frac{x}{U}$. Pro hac integrabilitate ponamus $y = ux$ et M et N functiones ipsius u; erit

$$\frac{dz}{z} = \frac{(M + Nu)dx + Nxdu}{Mx(\alpha + \beta u) + Nx(\gamma + \delta u)},$$

ubi integratio succedit sumendo $M = -Nu$, ut sit

$$\frac{dz}{z} = \frac{du}{\gamma + (\delta - \alpha)u - \beta uu} \quad \text{seu} \quad lV = \int \frac{du}{\gamma + (\delta - \alpha)u - \beta uu}$$

prorsus ut ante.

PROBLEMA 36

209. *Si posito* $dz = pdx + qdy$ *debeat esse* $Z = pP + qQ$ *existente* Z *functione ipsius* z *tantum,* P *et* Q *autem functionibus ipsarum* x *et* y *quibusvis datis, indolem functionis* z *investigare.*

SOLUTIO

Formentur sequentes aequationes ex propositis

$$Ldz = Lpdx + Lqdy,$$

$$MZdx = MpPdx + MqQdx, \quad NZdy = NpPdy + NqQdy,$$

quae in unam summam collectae dabunt

$$Ldz + Z(Mdx + Ndy) = p((L + MP)dx + NPdy) + q((L + NQ)dy + MQdx).$$

Ut iam pars posterior habeat factorem a litteris p et q liberum, fiat

$$L + MP : NP = MQ : L + NQ,$$

unde fit

$$LL + LNQ + LMP = 0 \quad \text{seu} \quad L = -MP - NQ,$$

quo valore inducto erit

$$-dz(MP + NQ) + Z(Mdx + Ndy) = (Mq - Np)(Qdx - Pdy).$$

Cum nunc P et Q sint functiones datae ipsarum x et y, dabitur multiplicator R, ut fiat

$$R(Qdx - Pdy) = dU$$

ideoque

$$-dz(MP + NQ) + Z(Mdx + Ndy) = \frac{Mq - Np}{R}dU.$$

Pro parte priori capiantur functiones indefinitae M et N ita, ut formula $\frac{Mdx + Ndy}{MP + NQ}$ integrabilis evadat, id quod semper fieri licet, sitque

$$\frac{Mdx + Ndy}{MP + NQ} = dV$$

et ob

$$Mdx + Ndy = (MP + NQ)dV$$

aequatio nostra hanc induet formam

$$(MP + NQ)(-dz + ZdV) = \frac{Mq - Np}{R}dU$$

seu

$$\frac{dz}{Z} - dV = \frac{Np - Mq}{RZ(MP + NQ)}dU.$$

Statuatur iam

$$\frac{Np - Mq}{RZ(MP + NQ)} = f' : U$$

atque habebitur

$$\int \frac{dz}{Z} - V = f : U \quad \text{seu} \quad \int \frac{dz}{Z} = V + f : U,$$

unde z determinatur per x et y.

COROLLARIUM 1

210. Pro solutione ergo problematis quaeratur primo ad formulam $Qdx - Pdy$ multiplicator R eam reddens integrabilem statuaturque

$$R(Qdx - Pdy) = dU,$$

unde colligitur quantitas U per binas variabiles x et y expressa.

COROLLARIUM 2

211. Deinde quantitates M et N ita capiantur, ut formula $\frac{Mdx + Ndy}{MP + NQ}$ fiat integrabilis; cuius integrale si statuatur $= V$, statim habetur solutio generalis problematis, quae dat

$$\int \frac{dz}{Z} = V + f: U.$$

EXEMPLUM

212. *Si P et Q sint functiones homogeneae ipsarum x et y, utraque dimensionum numeri $= n$, solutionem problematis perficere.*

Ponatur $y = ux$ et tam P quam Q fiet productum ex potestate x^n in functionem quandam ipsius u. Sit ergo

$$P = x^n S \quad \text{et} \quad Q = x^n T$$

eruntque S et T functiones datae ipsius u. Tum vero ob $dy = udx + xdu$ formula $Qdx - Pdy$ abit in

$$x^n Tdx - x^n Sudx - x^{n+1} Sdu = x^n((T - Su)dx - Sxdu).$$

Sumatur ergo

$$R = \frac{1}{x^{n+1}(T - Su)}$$

fietque

$$dU = \frac{dx}{x} - \frac{Sdu}{T - Su},$$

unde colligitur U.

Deinde pro altera quantitate V habebimus hanc aequationem

$$dV = \frac{(M+Nu)dx + Nxdu}{x^n(MS+NT)},$$

ubi iam facile est pro M et N eiusmodi functiones ipsius u assumere, ut haec formula integrationem admittat. Integrale scilicet erit

$$V = \frac{-M-Nu}{(n-1)x^{n-1}(MS+NT)},$$

at M et N seu $\frac{M}{N} = K$ ita accipi debet, ut fiat

$$-\frac{1}{(n-1)x^{n-1}} d.\frac{K+u}{KS+T} = \frac{1}{x^{n-1}} \cdot \frac{du}{KS+T}$$

seu

$$\begin{gathered}-KKdS + KSdu - uKdS - uSdK + TdK - KdT + Tdu - udT \\ + (n-1)du(KS+T) = 0,\end{gathered}$$

quae ad hanc formam reducitur

$$(T-Su)dK + K(nSdu - udS - dT) - KKdS + nTdu - udT = 0.$$

Ex qua concessa aequationum resolutione cognoscitur quantitas K, qua inventa erit

$$V = \frac{-K-u}{(n-1)x^{n-1}(KS+T)}.$$

Cum autem illa aequatio solutu difficilis videatur, ponatur statim

$$\frac{K+u}{KS+T} = v$$

eritque

$$K = \frac{Tv-u}{1-Sv} \quad \text{et} \quad KS+T = \frac{T-Su}{1-Sv},$$

unde fit

$$dv + \frac{(n-1)du(1-Sv)}{T-Su} = 0,$$

qua resoluta erit

$$V = \frac{-v}{(n-1)x^{n-1}}.$$

COROLLARIUM

213. Casus autem, quo $n=1$, singulari evolutione eget. Facile autem patet tum sumi debere $M=-Nu$, ut fiat $dV=\frac{du}{T-Su}$; unde, postquam quantitas V fuerit inventa, erit semper

$$\int\frac{dz}{Z}=V+f:U.$$

SCHOLION

214. Cum ternae variabiles x, y, z sint inter se permutabiles, patet hoc problema multo latius extendi posse. Scilicet si conditio proposita hac contineatur aequatione $pP+qQ+R=0$, non solum solvendi methodus adhibita succedit, si R sit functio ipsius z et P cum Q functiones ipsarum x et y, sed etiam, si fuerit P functio ipsius x et Q et R functiones ipsarum y et z, tum vero etiam, si Q functio ipsius y, at P et R functiones binarum reliquarum x et z. Haec vero conditio cum ante tractatis eo redit, ut binae formulae differentiales p et q sint a se invicem separatae neque plus una dimensione occupent, etiamsi et his casibus ingens restrictio accedat. Quodsi autem conditio magis sit complicata, solutio vix unquam sperari posse videtur; interim tamen casum eiusmodi proferam, quo solutionem expedire licet.

PROBLEMA 37

215. *Si posito $dz=pdx+qdy$ debeat esse $q=Ap^n x^\lambda y^\mu z^\nu$, indolem functionis z in genere investigare.*

SOLUTIO

Posito hoc valore loco q habebimus

$$dz=pdx+Ap^n x^\lambda y^\mu z^\nu dy,$$

unde fit

$$Ay^\mu dy=p^{-n}x^{-\lambda}z^{-\nu}(dz-pdx).$$

Ponatur $p^{-n}x^{-\lambda}z^{-\nu}=t$, ut sit $p=t^{-\frac{1}{n}}x^{-\frac{\lambda}{n}}z^{-\frac{\nu}{n}}$, eritque

$$Ay^\mu dy=tdz-t^{\frac{n-1}{n}}x^{-\frac{\lambda}{n}}z^{-\frac{\nu}{n}}dx.$$

Statuatur porro $t^{n-1}z^{-\nu}=u^n$ seu $t=z^{\frac{\nu}{n-1}}u^{\frac{n}{n-1}}$; erit

$$Ay^\mu dy = u^{\frac{n}{n-1}}z^{\frac{\nu}{n-1}}dz - ux^{-\frac{\lambda}{n}}dx.$$

Iam partibus, quoad fieri licet, integratis adipiscimur

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{n-1}{n+\nu-1}u^{\frac{n}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{nu}{n-\lambda}x^{\frac{n-\lambda}{n}}-\int du\left(\frac{n}{n+\nu-1}u^{\frac{1}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{n}{n-\lambda}x^{\frac{n-\lambda}{n}}\right)$$

ac nunc solutionem per praecepta supra data expedire licet; scilicet statuatur

$$\frac{1}{n+\nu-1}u^{\frac{1}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{1}{n-\lambda}x^{\frac{n-\lambda}{n}}=f':u$$

eritque

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{n-1}{n+\nu-1}u^{\frac{n}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{n}{n-\lambda}ux^{\frac{n-\lambda}{n}}-nf:u,$$

atque ex his binis aequationibus si elidatur u, dabitur utique z per x et y.

COROLLARIUM 1

216. Casus $n=1$ peculiarem postulat tractationem; cum enim posito $p=\frac{1}{t}x^{-\lambda}z^{-\nu}$ sit $Ay^\mu dy=tdz-x^{-\lambda}z^{-\nu}dx$, erit

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{1}{\lambda-1}x^{1-\lambda}z^{-\nu}+\int dz\left(t+\frac{\nu}{\lambda-1}x^{1-\lambda}z^{-\nu-1}\right)$$

atque hinc statim concluditur

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{1}{\lambda-1}x^{1-\lambda}z^{-\nu}+f:z.$$

COROLLARIUM 2

217. Casus autem $n+\nu-1=0$ et $n-\lambda=0$ nullam facessunt molestiam, cum sit priori casu

$$\frac{n-1}{n+\nu-1}z^{\frac{n+\nu-1}{n-1}}=lz,$$

posteriori autem

$$\frac{n}{n-\lambda}x^{\frac{n-\lambda}{n}}=lx,$$

quos valores in solutionem introduci oportet.

EXEMPLUM

218. *Si posito* $dz = pdx + qdy$ *debeat esse* $pqxy = az$ *seu* $q = \frac{az}{pxy}$, *functionem* z *investigare.*

Erit ergo

$$dz = pdx + \frac{azdy}{pxy} \quad \text{seu} \quad \frac{ady}{y} = \frac{px}{z}(dz - pdx).$$

Ponatur $\frac{px}{z} = t$ seu $p = \frac{tz}{x}$; erit

$$\frac{ady}{y} = tdz - \frac{ttzdx}{x}.$$

Statuatur porro $ttz = uu$ seu $t = \frac{u}{\sqrt{z}}$, ut sit

$$\frac{ady}{y} = \frac{udz}{\sqrt{z}} - \frac{uudx}{x}$$

et, quoad fieri potest, integrando

$$aly = 2u\sqrt{z} - uulx - \int du(2\sqrt{z} - 2ulx),$$

ita ut iam post signum integrale unicum differentiale du reperiatur. Posito ergo

$$\sqrt{z} - ulx = f' : u$$

erit

$$aly = 2u\sqrt{z} - uulx - 2f:u = uulx + 2uf':u - 2f:u.$$

Pro casu simplicissimo sumatur $f':u = 0$ et $f:u = 0$; erit $u = \frac{\sqrt{z}}{lx}$ ideoque

$$aly = \frac{2z}{lx} - \frac{z}{lx} = \frac{z}{lx},$$

ita ut pro casu simplicissimo sit $z = alx \cdot ly$.

Si ponatur $f':u = ulc$ et $f:u = \frac{1}{2}uulc$, erit

$$u = \frac{\sqrt{z}}{lx + lc} = \frac{\sqrt{z}}{lcx} \quad \text{et} \quad aly = \frac{2z}{lcx} - \frac{zlx}{(lcx)^2} - \frac{zlc}{(lcx)^2} = \frac{z}{lcx},$$

ita ut sit $z = aly(lc + lx)$; magis generaliter autem erit

$$z = a(lb + ly)(lc + lx).$$

Statuatur porro $t^{n-1}z^{-\nu}=u^n$ seu $t=z^{\frac{\nu}{n-1}}u^{\frac{n}{n-1}}$; erit

$$Ay^{\mu}dy=u^{\frac{n}{n-1}}z^{\frac{\nu}{n-1}}dz-ux^{-\frac{\lambda}{n}}dx.$$

Iam partibus, quoad fieri licet, integratis adipiscimur

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{n-1}{n+\nu-1}u^{\frac{n}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{nu}{n-\lambda}x^{\frac{n-\lambda}{n}}-\int du\left(\frac{n}{n+\nu-1}u^{\frac{1}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{n}{n-\lambda}x^{\frac{n-\lambda}{n}}\right)$$

ac nunc solutionem per praecepta supra data expedire licet; scilicet statuatur

$$\frac{1}{n+\nu-1}u^{\frac{1}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{1}{n-\lambda}x^{\frac{n-\lambda}{n}}=f':u$$

eritque

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{n-1}{n+\nu-1}u^{\frac{n}{n-1}}z^{\frac{n+\nu-1}{n-1}}-\frac{n}{n-\lambda}ux^{\frac{n-\lambda}{n}}-nf:u,$$

atque ex his binis aequationibus si elidatur u, dabitur utique z per x et y.

COROLLARIUM 1

216. Casus $n=1$ peculiarem postulat tractationem; cum enim posito $p=\frac{1}{t}x^{-\lambda}z^{-\nu}$ sit $Ay^{\mu}dy=tdz-x^{-\lambda}z^{-\nu}dx$, erit

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{1}{\lambda-1}x^{1-\lambda}z^{-\nu}+\int dz\left(t+\frac{\nu}{\lambda-1}x^{1-\lambda}z^{-\nu-1}\right)$$

atque hinc statim concluditur

$$\frac{A}{\mu+1}y^{\mu+1}=\frac{1}{\lambda-1}x^{1-\lambda}z^{-\nu}+f:z.$$

COROLLARIUM 2

217. Casus autem $n+\nu-1=0$ et $n-\lambda=0$ nullam facessunt molestiam, cum sit priori casu

$$\frac{n-1}{n+\nu-1}z^{\frac{n+\nu-1}{n-1}}=lz,$$

posteriori autem

$$\frac{n}{n-\lambda}x^{\frac{n-\lambda}{n}}=lx,$$

quos valores in solutionem introduci oportet.

EXEMPLUM

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Erit ergo

$$dz = pdx + \frac{azdy}{pxy} \quad \text{seu} \quad \frac{ady}{y} = \frac{px}{z}(dz - pdx).$$

Ponatur $\frac{px}{z} = t$ seu $p = \frac{tz}{x}$; erit

$$\frac{ady}{y} = tdz - \frac{ttzdx}{x}.$$

Statuatur porro $ttz = uu$ seu $t = \frac{u}{\sqrt{z}}$, ut sit

$$\frac{ady}{y} = \frac{udz}{\sqrt{z}} - \frac{uudx}{x}$$

et, quoad fieri potest, integrando

$$aly = 2u\sqrt{z} - uulx - \int du(2\sqrt{z} - 2ulx),$$

ita ut iam post signum integrale unicum differentiale du reperiatur. Posito ergo

$$\sqrt{z} - ulx = f' : u$$

erit

$$aly = 2u\sqrt{z} - uulx - 2f : u = uulx + 2uf' : u - 2f : u.$$

Pro casu simplicissimo sumatur $f' : u = 0$ et $f : u = 0$; erit $u = \frac{\sqrt{z}}{lx}$ ideoque

$$aly = \frac{2z}{lx} - \frac{z}{lx} = \frac{z}{lx},$$

ita ut pro casu simplicissimo sit $z = alx \cdot ly$.

Si ponatur $f' : u = ulc$ et $f : u = \frac{1}{2}uulc$, erit

$$u = \frac{\sqrt{z}}{lx + lc} = \frac{\sqrt{z}}{lcx} \quad \text{et} \quad aly = \frac{2z}{lcx} - \frac{zlx}{(lcx)^2} - \frac{zlc}{(lcx)^2} = \frac{z}{lcx},$$

ita ut sit $z = aly(lc + lx)$; magis generaliter autem erit

$$z = a(lb + ly)(lc + lx).$$

SCHOLION

219. Methodi hactenus traditae haud mediocriter amplificabuntur, si loco binarum variabilium x et y, quarum functio esse debet z, binae aliae variabiles t et u introducantur, quarum relatio ad illas detur. Ita si z sit functio binarum variabilium x et y, ut inde prodeat

$$dz = pdx + qdy,$$

ac loco x et y aliae novae variabiles t et u introducantur, ut iam differentiatione instituta prodeat

$$dz = rdt + sdu,$$

quaeritur, quomodo r et s per p et q determinentur pro relatione inter pristinas variabiles x, y et novas t et u stabilita. Hinc ergo tam x quam y certae cuidam functioni ipsarum t et u aequabitur; quae cum detur, sit

$$dx = Pdt + Qdu \quad \text{et} \quad dy = Rdt + Sdu,$$

ita ut facta hac substitutione z iam sit functio ipsarum t et u. Cum igitur esset $dz = pdx + qdy$, erit nunc

$$dz = (Pp + Rq)dt + (Qp + Sq)du.$$

Est vero per hypothesin $dz = rdt + sdu$, unde habebitur

$$r = Pp + Rq \quad \text{et} \quad s = Qp + Sq.$$

Quare facta hac substitutione valores differentiales novi ex praecedentibus ita determinabuntur, ut sit

$$\left(\frac{dz}{dt}\right) = P\left(\frac{dz}{dx}\right) + R\left(\frac{dz}{dy}\right) \quad \text{et} \quad \left(\frac{dz}{du}\right) = Q\left(\frac{dz}{dx}\right) + S\left(\frac{dz}{dy}\right).$$

Unde etiam, cum sit vicissim

$$Qr - Ps = (QR - PS)q \quad \text{et} \quad Sr - Rs = (PS - QR)p,$$

concludimus fore

$$\left(\frac{dz}{dx}\right) = \frac{S}{PS - QR}\left(\frac{dz}{dt}\right) - \frac{R}{PS - QR}\left(\frac{dz}{du}\right)$$

et

$$\left(\frac{dz}{dy}\right) = \frac{-Q}{PS - QR}\left(\frac{dz}{dt}\right) + \frac{P}{PS - QR}\left(\frac{dz}{du}\right).$$

Vel cum x et y perinde ac z sint functiones ipsarum t et u, haec relatio ita exprimi potest, ut sit

$$\left(\frac{dz}{dt}\right)=\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right) \quad \text{et} \quad \left(\frac{dz}{du}\right)=\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Hinc efficitur, ut, quae problemata pro data quadam relatione inter p, q, x, y, z resolvi possunt, ea quoque pro relatione inde resultante inter r, s, t, u et z resolvi queant; unde saepe problemata nascuntur, quae solutu vehementer difficilia videantur, ex quo non contemnenda subsidia in hanc Analyseos partem inferri possent; sed quia usus praecipue in formulis differentialibus secundi gradus spectatur, his non fusius immorans ad eas evolvendas progredior.[1])

1) Notandum est EULERUM nonnullas aequationes differentiales eiusdem generis, quod hic in Capitibus II—VI tractatur, iam resolvisse in Commentatione 285 (indicis ENESTROEMIANI): *Investigatio functionum ex data differentialium conditione*, Novi comment. acad. sc. Petrop. 9 (1762/3), 1764, p. 170; *LEONHARDI EULERI Opera omnia*, series I, vol. 22. F. E.

CALCVLI INTEGRALIS
LIBER POSTERIOR.

PARS PRIMA

SEV

INVESTIGATIO FVNCTIONVM DVARVM VARIABILIVM EX DATA DIFFERENTIALIVM CVIVSVIS GRADVS RELATIONE.

SECTIO SECVNDA

INVESTIGATIO DVARVM VARIABILIVM FVNCTIONVM EX DATA DIFFERENTIALIVM SECVNDI GRADVS RELATIONE.

CAPUT I

DE FORMULIS DIFFERENTIALIBUS SECUNDI GRADUS IN GENERE

PROBLEMA 38

220. *Si z sit functio quaecunque binarum variabilium x et y, eius formulas differentiales secundi gradus exhibere.*

SOLUTIO

Cum z sit functio binarum variabilium x et y, eius differentiale huiusmodi habebit formam $dz = pdx + qdy$, ex qua p et q sunt formulae differentiales primi gradus, quas ita denotare solemus

$$p = \left(\frac{dz}{dx}\right) \quad \text{et} \quad q = \left(\frac{dz}{dy}\right).$$

Cum nunc sint quoque p et q functiones ipsarum x et y, formulae differentiales inde natae erunt formulae differentiales secundi gradus ipsius z, unde intelligitur quatuor huiusmodi formulas nasci

$$\left(\frac{dp}{dx}\right), \quad \left(\frac{dp}{dy}\right), \quad \left(\frac{dq}{dx}\right), \quad \left(\frac{dq}{dy}\right),$$

quarum autem secundam ac tertiam inter se congruere in Calculo differentiali est demonstratum.[1]) Sed cum sit $p = \left(\frac{dz}{dx}\right)$, simili scribendi ratione erit

1) *Institutionum calculi differentialis* partis prioris § 228 et quae sequuntur; *Leonhardi Euleri Opera omnia,* series I, vol. 10, p. 154. F. E.

$\left(\frac{dp}{dx}\right) = \left(\frac{ddz}{dx^2}\right)$, cuius scripturae significatus hinc sponte patet. Deinde eodem modo erit $\left(\frac{dp}{dy}\right) = \left(\frac{ddz}{dxdy}\right)$ atque ob $q = \left(\frac{dz}{dy}\right)$ habebimus $\left(\frac{dq}{dx}\right) = \left(\frac{ddz}{dydx}\right)$ et $\left(\frac{dq}{dy}\right) = \left(\frac{ddz}{dy^2}\right)$. Quia ergo est

$$\left(\frac{ddz}{dydx}\right) = \left(\frac{ddz}{dxdy}\right),$$

functioni z convenient tres formulae differentiales secundi gradus, quae sunt

$$\left(\frac{ddz}{dx^2}\right), \quad \left(\frac{ddz}{dxdy}\right) \quad \text{et} \quad \left(\frac{ddz}{dy^2}\right).$$

COROLLARIUM 1

221. Ut ergo functio z duarum variabilium x et y duas habet formulas differentiales primi gradus

$$\left(\frac{dz}{dx}\right) \quad \text{et} \quad \left(\frac{dz}{dy}\right),$$

ita habet tres formulas differentiales secundi gradus

$$\left(\frac{ddz}{dx^2}\right), \quad \left(\frac{ddz}{dxdy}\right) \quad \text{et} \quad \left(\frac{ddz}{dy^2}\right).$$

COROLLARIUM 2

222. Hae ergo formulae per duplicem differentiationem nascuntur unicam tantum quantitatem pro variabili accipiendo. In prima scilicet bis eadem x variabilis sumitur, in secunda vero in altera differentiatione x, in altera autem y variabilis accipitur, in tertia autem bis y.

COROLLARIUM 3

223. Simili modo patet eiusdem functionis z quatuor dari formulas differentiales tertii gradus, scilicet

$$\left(\frac{d^3z}{dz^3}\right), \quad \left(\frac{d^3z}{dx^2dy}\right), \quad \left(\frac{d^3z}{dxdy^2}\right), \quad \left(\frac{d^3z}{dy^3}\right),$$

quarti autem gradus quinque, quinti sex etc.

SCHOLION

224. Formulae hae differentiales secundi gradus ope substitutionis saltem ad formam primi gradus revocari possunt. Veluti formula $\left(\frac{ddz}{dx^2}\right)$, si ponatur $\left(\frac{dz}{dx}\right)=p$, transformabitur in $\left(\frac{dp}{dx}\right)$, formula autem $\left(\frac{ddz}{dxdy}\right)$ eadem substitutione in hanc $\left(\frac{dp}{dy}\right)$. At posito $\left(\frac{dz}{dy}\right)=q$ formula $\left(\frac{ddz}{dxdy}\right)$ transmutatur in hanc $\left(\frac{dq}{dx}\right)$, formula autem $\left(\frac{ddz}{dy^2}\right)$ in hanc $\left(\frac{dq}{dy}\right)$. Vicissim autem uti ex aequalitate $p=\left(\frac{dz}{dx}\right)$ deduximus

$$\left(\frac{dp}{dx}\right)=\left(\frac{ddz}{dx^2}\right) \quad \text{et} \quad \left(\frac{dp}{dy}\right)=\left(\frac{ddz}{dxdy}\right),$$

ita ex his ulterius progrediendo colligemus

$$\left(\frac{ddp}{dx^2}\right)=\left(\frac{d^3z}{dx^3}\right),\quad \left(\frac{ddp}{dxdy}\right)=\left(\frac{d^3z}{dx^2dy}\right),\quad \left(\frac{ddp}{dy^2}\right)=\left(\frac{d^3z}{dxdy^2}\right).$$

Tum vero etiam si ponamus $\left(\frac{dp}{dx}\right)=\left(\frac{dz}{dy}\right)$, hinc sequentur istae aequalitates $\left(\frac{ddp}{dx^2}\right)=\left(\frac{ddz}{dxdy}\right)$ et $\left(\frac{ddp}{dxdy}\right)=\left(\frac{ddz}{dy^2}\right)$. Hicque est quasi novus algorithmus, cuius principia per se ita sunt manifesta, ut maiore illustratione non indigeant.

EXEMPLUM 1

225. *Si sit $z=xy$, eius formulas differentiales secundi gradus exhibere.*

Cum sit

$$\left(\frac{dz}{dx}\right)=y \quad \text{et} \quad \left(\frac{dz}{dy}\right)=x,$$

erit

$$\left(\frac{ddz}{dx^2}\right)=0,\quad \left(\frac{ddz}{dxdy}\right)=1 \quad \text{et} \quad \left(\frac{ddz}{dy^2}\right)=0.$$

EXEMPLUM 2

226. *Si sit $z=x^my^n$, eius formulas differentiales secundi gradus exhibere.*

Cum sit

$$\left(\frac{dz}{dx}\right)=mx^{m-1}y^n \quad \text{et} \quad \left(\frac{dz}{dy}\right)=nx^my^{n-1},$$

erit

$$\left(\frac{ddz}{dx^2}\right)=m(m-1)x^{m-2}y^n,\quad \left(\frac{ddz}{dxdy}\right)=mnx^{m-1}y^{n-1},\quad \left(\frac{ddz}{dy^2}\right)=n(n-1)x^my^{n-2}.$$

EXEMPLUM 3

227. *Si sit* $z = \sqrt{(xx+yy)}$, *eius formulas differentiales secundi gradus exhibere.*

Cum sit

$$\left(\frac{dz}{dx}\right) = \frac{x}{\sqrt{(xx+yy)}} \quad \text{et} \quad \left(\frac{dz}{dy}\right) = \frac{y}{\sqrt{(xx+yy)}},$$

erit

$$\left(\frac{ddz}{dx^2}\right) = \frac{yy}{(xx+yy)^{\frac{3}{2}}}, \quad \left(\frac{ddz}{dxdy}\right) = \frac{-xy}{(xx+yy)^{\frac{3}{2}}}, \quad \left(\frac{ddz}{dy^2}\right) = \frac{xx}{(xx+yy)^{\frac{3}{2}}}.$$

SCHOLION

228. Quemadmodum binae formulae differentiales primi gradus cuiusque functionis z ita sunt comparatae, ut sit

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right)$$

et integrando

$$z = \int \left(dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right)\right),$$

ita quoque in formulis secundi gradus erit

$$\left(\frac{dz}{dx}\right) = \int \left(dx\left(\frac{ddz}{dx^2}\right) + dy\left(\frac{ddz}{dxdy}\right)\right) \quad \text{et} \quad \left(\frac{dz}{dy}\right) = \int \left(dx\left(\frac{ddz}{dxdy}\right) + dy\left(\frac{ddz}{dy^2}\right)\right).$$

Tres igitur formulae secundi gradus semper ita sunt comparatae, ut geminam integrationem praebeant, si scilicet cum differentialibus dx et dy rite combinentur; haecque proprietas, quae probe notetur, in sequentibus insigne adiumentum afferet.

PROBLEMA 39

229. *Si z sit functio binarum variabilium x et y, loco x et y introducantur binae novae variabiles t et u, ita ut tam x quam y aequetur certae functioni ipsarum t et u; formulas differentiales secundi gradus ipsius z respectu harum novarum variabilium definire.*

SOLUTIO

Quatenus z per x et y datur, datae sunt eius formulae differentiales tam primi gradus $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ quam secundi gradus $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$, $\left(\frac{ddz}{dy^2}\right)$; ex quibus

quomodo formulae differentiales respectu novarum variabilium t et u determinentur, definiri oportet.

Pro primo gradu autem cum sit

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right),$$

quia tam x quam y datur per t et u, erit

$$dx = dt\left(\frac{dx}{dt}\right) + du\left(\frac{dx}{du}\right) \quad \text{et} \quad dy = dt\left(\frac{dy}{dt}\right) + du\left(\frac{dy}{du}\right),$$

quibus valoribus substitutis habebitur ipsius z differentiale plenum ex variatione utriusque t et u ortum

$$dz = dt\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right) + du\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right) + dt\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right) + du\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Quodsi iam vel sola t variabilis sumatur vel sola u, prodibunt formulae differentiales primi gradus

$$\left(\frac{dz}{dt}\right) = \left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right) + \left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right), \quad \left(\frac{dz}{du}\right) = \left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right) + \left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Simili modo ulterius progrediendo differentiemus formulas

$$\left(\frac{dz}{dx}\right) = p \quad \text{et} \quad \left(\frac{dz}{dy}\right) = q$$

primo generaliter, tum vero loco x et y etiam t et u introducamus; hincque nanciscemur

$$\left(\frac{dp}{dt}\right) = \left(\frac{dx}{dt}\right)\left(\frac{dp}{dx}\right) + \left(\frac{dy}{dt}\right)\left(\frac{dp}{dy}\right), \quad \left(\frac{dp}{du}\right) = \left(\frac{dx}{du}\right)\left(\frac{dp}{dx}\right) + \left(\frac{dy}{du}\right)\left(\frac{dp}{dy}\right),$$

$$\left(\frac{dq}{dt}\right) = \left(\frac{dx}{dt}\right)\left(\frac{dq}{dx}\right) + \left(\frac{dy}{dt}\right)\left(\frac{dq}{dy}\right), \quad \left(\frac{dq}{du}\right) = \left(\frac{dx}{du}\right)\left(\frac{dq}{dx}\right) + \left(\frac{dy}{du}\right)\left(\frac{dq}{dy}\right).$$

Unde poterimus formulas [differentiales quantitatum] $\left(\frac{dz}{dt}\right)$ et $\left(\frac{dz}{du}\right)$ pro variabilitate tam solius t quam solius u assignare; scilicet cum sit

$$\left(\frac{dz}{dt}\right) = p\left(\frac{dx}{dt}\right) + q\left(\frac{dy}{dt}\right) \quad \text{et} \quad \left(\frac{dz}{du}\right) = p\left(\frac{dx}{du}\right) + q\left(\frac{dy}{du}\right),$$

EXEMPLUM 3

227. *Si sit $z = \sqrt{(xx + yy)}$, eius formulas differentiales secundi gradus exhibere.*

Cum sit

$$\left(\frac{dz}{dx}\right) = \frac{x}{\sqrt{(xx+yy)}} \quad \text{et} \quad \left(\frac{dz}{dy}\right) = \frac{y}{\sqrt{(xx+yy)}},$$

erit

$$\left(\frac{ddz}{dx^2}\right) = \frac{yy}{(xx+yy)^{\frac{3}{2}}}, \quad \left(\frac{ddz}{dxdy}\right) = \frac{-xy}{(xx+yy)^{\frac{3}{2}}}, \quad \left(\frac{ddz}{dy^2}\right) = \frac{xx}{(xx+yy)^{\frac{3}{2}}}.$$

SCHOLION

228. Quemadmodum binae formulae differentiales primi gradus cuiusque functionis z ita sunt comparatae, ut sit

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right)$$

et integrando

$$z = \int\left(dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right)\right),$$

ita quoque in formulis secundi gradus erit

$$\left(\frac{dz}{dx}\right) = \int\left(dx\left(\frac{ddz}{dx^2}\right) + dy\left(\frac{ddz}{dxdy}\right)\right) \quad \text{et} \quad \left(\frac{dz}{dy}\right) = \int\left(dx\left(\frac{ddz}{dxdy}\right) + dy\left(\frac{ddz}{dy^2}\right)\right).$$

Tres igitur formulae secundi gradus semper ita sunt comparatae, ut geminam integrationem praebeant, si scilicet cum differentialibus dx et dy rite combinentur; haecque proprietas, quae probe notetur, in sequentibus insigne adiumentum afferet.

PROBLEMA 39

229. *Si z sit functio binarum variabilium x et y, loco x et y introducantur binae novae variabiles t et u, ita ut tam x quam y aequetur certae functioni ipsarum t et u; formulas differentiales secundi gradus ipsius z respectu harum novarum variabilium definire.*

SOLUTIO

Quatenus z per x et y datur, datae sunt eius formulae differentiales tam primi gradus $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ quam secundi gradus $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$, $\left(\frac{ddz}{dy^2}\right)$; ex quibus

quomodo formulae differentiales respectu novarum variabilium t et u determinentur, definiri oportet.

Pro primo gradu autem cum sit

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right),$$

quia tam x quam y datur per t et u, erit

$$dx = dt\left(\frac{dx}{dt}\right) + du\left(\frac{dx}{du}\right) \quad \text{et} \quad dy = dt\left(\frac{dy}{dt}\right) + du\left(\frac{dy}{du}\right),$$

quibus valoribus substitutis habebitur ipsius z differentiale plenum ex variatione utriusque t et u ortum

$$dz = dt\left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right) + du\left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right) + dt\left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right) + du\left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Quodsi iam vel sola t variabilis sumatur vel sola u, prodibunt formulae differentiales primi gradus

$$\left(\frac{dz}{dt}\right) = \left(\frac{dx}{dt}\right)\left(\frac{dz}{dx}\right) + \left(\frac{dy}{dt}\right)\left(\frac{dz}{dy}\right), \quad \left(\frac{dz}{du}\right) = \left(\frac{dx}{du}\right)\left(\frac{dz}{dx}\right) + \left(\frac{dy}{du}\right)\left(\frac{dz}{dy}\right).$$

Simili modo ulterius progrediendo differentiemus formulas

$$\left(\frac{dz}{dx}\right) = p \quad \text{et} \quad \left(\frac{dz}{dy}\right) = q$$

primo generaliter, tum vero loco x et y etiam t et u introducamus; hincque nanciscemur

$$\left(\frac{dp}{dt}\right) = \left(\frac{dx}{dt}\right)\left(\frac{dp}{dx}\right) + \left(\frac{dy}{dt}\right)\left(\frac{dp}{dy}\right), \quad \left(\frac{dp}{du}\right) = \left(\frac{dx}{du}\right)\left(\frac{dp}{dx}\right) + \left(\frac{dy}{du}\right)\left(\frac{dp}{dy}\right),$$

$$\left(\frac{dq}{dt}\right) = \left(\frac{dx}{dt}\right)\left(\frac{dq}{dx}\right) + \left(\frac{dy}{dt}\right)\left(\frac{dq}{dy}\right), \quad \left(\frac{dq}{du}\right) = \left(\frac{dx}{du}\right)\left(\frac{dq}{dx}\right) + \left(\frac{dy}{du}\right)\left(\frac{dq}{dy}\right).$$

Unde poterimus formulas [differentiales quantitatum] $\left(\frac{dz}{dt}\right)$ et $\left(\frac{dz}{du}\right)$ pro variabilitate tam solius t quam solius u assignare; scilicet cum sit

$$\left(\frac{dz}{dt}\right) = p\left(\frac{dx}{dt}\right) + q\left(\frac{dy}{dt}\right) \quad \text{et} \quad \left(\frac{dz}{du}\right) = p\left(\frac{dx}{du}\right) + q\left(\frac{dy}{du}\right),$$

inveniemus

$$\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddx}{dt^2}\right)\left(\frac{dz}{dx}\right) + \left(\frac{ddy}{dt^2}\right)\left(\frac{dz}{dy}\right)$$

$$+ \left(\frac{dx}{dt}\right)^2\left(\frac{ddz}{dx^2}\right) + 2\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right)\left(\frac{ddz}{dxdy}\right) + \left(\frac{dy}{dt}\right)^2\left(\frac{ddz}{dy^2}\right),$$

$$\left(\frac{ddz}{dtdu}\right) = \left(\frac{ddx}{dtdu}\right)\left(\frac{dz}{dx}\right) + \left(\frac{ddy}{dtdu}\right)\left(\frac{dz}{dy}\right)$$

$$+ \left(\frac{dx}{dt}\right)\left(\frac{dx}{du}\right)\left(\frac{ddz}{dx^2}\right) + \left(\frac{dx}{dt}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dxdy}\right) + \left(\frac{dy}{dt}\right)\left(\frac{dx}{du}\right)\left(\frac{ddz}{dxdy}\right) + \left(\frac{dy}{dt}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dy^2}\right),$$

$$\left(\frac{ddz}{du^2}\right) = \left(\frac{ddx}{du^2}\right)\left(\frac{dz}{dx}\right) + \left(\frac{ddy}{du^2}\right)\left(\frac{dz}{dy}\right)$$

$$+ \left(\frac{dx}{du}\right)^2\left(\frac{ddz}{dx^2}\right) + 2\left(\frac{dx}{du}\right)\left(\frac{dy}{du}\right)\left(\frac{ddz}{dxdy}\right) + \left(\frac{dy}{du}\right)^2\left(\frac{ddz}{dy^2}\right).$$

COROLLARIUM 1

230. Proposita ergo conditione quadam inter formulas differentiales functionis z, quatenus per variabiles t et u definitur, eadem conditio pro eadem functione z transfertur ad alias binas variabiles x et y ab illis utcunque pendentes.

COROLLARIUM 2

231. Formulae quidem

$$\left(\frac{dx}{dt}\right), \quad \left(\frac{dy}{dt}\right), \quad \left(\frac{dx}{du}\right), \quad \left(\frac{dy}{du}\right) \quad \text{etc.}$$

per t et u exprimuntur ex relatione, quae inter x, y et t, u assumitur, verum indidem eaedem formulae ad variabiles x et y revocari possunt.

SCHOLION

232. Quemadmodum hic variabilitas quantitatum t et u per formulas differentiales ex variabilibus x et y natas est expressa, ita vicissim, si variabiles t et u proponantur, ex quibus certo modo alterae x et y determinentur, sequentes reductiones habebuntur facta tantum variabilium permutatione. Primo scilicet pro formulis primi gradus

$$\left(\frac{dz}{dx}\right) = \left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right) + \left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right), \quad \left(\frac{dz}{dy}\right) = \left(\frac{dt}{dy}\right)\left(\frac{dz}{dt}\right) + \left(\frac{du}{dy}\right)\left(\frac{dz}{du}\right);$$

pro formulis autem differentialibus secundi gradus

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddt}{dx^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dx^2}\right)\left(\frac{dz}{du}\right)$$
$$+\left(\frac{dt}{dx}\right)^2\left(\frac{ddz}{dt^2}\right)+2\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)^2\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right)=\left(\frac{ddt}{dxdy}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dxdy}\right)\left(\frac{dz}{du}\right)$$
$$+\left(\frac{dt}{dx}\right)\left(\frac{dt}{dy}\right)\left(\frac{ddz}{dt^2}\right)+\left(\frac{dt}{dx}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)\left(\frac{dt}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right)$$
$$+\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right)+2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right),$$

ubi determinatio litterarum t et u per alteras x et y considerari debet. Quoniam scilicet in conditionibus praescriptis binis variabilibus x et y uti solemus, earum loco alias quascunque t et u introducendo loco illarum formularum differentialium has novas formas ad variabiles t et u relatas adhibere poterimus, ubi deinceps relatio inter variabiles x, y et t, u ita est constituenda, ut quaestio solutu facilior evadat. Pro variis igitur huiusmodi relationibus exempla evolvamus.

EXEMPLUM 1

233. *Si inter variabiles x, y et t, u haec relatio constituatur, ut sit*

$$t=\alpha x+\beta y \quad et \quad u=\gamma x+\delta y,$$

reductionem formularum differentialium exhibere.

Cum sit

$$\left(\frac{dt}{dx}\right)=\alpha,\quad \left(\frac{dt}{dy}\right)=\beta,\quad \left(\frac{du}{dx}\right)=\gamma,\quad \left(\frac{du}{dy}\right)=\delta$$

hincque formulae pro secundo gradu evanescant, habebimus pro formulis primi gradus

$$\left(\frac{dz}{dx}\right)=\alpha\left(\frac{dz}{dt}\right)+\gamma\left(\frac{dz}{du}\right),\quad \left(\frac{dz}{dy}\right)=\beta\left(\frac{dz}{dt}\right)+\delta\left(\frac{dz}{du}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{ddz}{dx^2}\right) = \alpha\alpha\left(\frac{ddz}{dt^2}\right) + 2\alpha\gamma\left(\frac{ddz}{dt\,du}\right) + \gamma\gamma\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dx\,dy}\right) = \alpha\beta\left(\frac{ddz}{dt^2}\right) + (\alpha\delta + \beta\gamma)\left(\frac{ddz}{dt\,du}\right) + \gamma\delta\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right) = \beta\beta\left(\frac{ddz}{dt^2}\right) + 2\beta\delta\left(\frac{ddz}{dt\,du}\right) + \delta\delta\left(\frac{ddz}{du^2}\right).$$

COROLLARIUM 1

234. Si sumatur $t = x$ et $u = x + y$, erit

$$\alpha = 1, \quad \beta = 0, \quad \gamma = 1 \quad \text{et} \quad \delta = 1;$$

erit

$$\left(\frac{dz}{dx}\right) = \left(\frac{dz}{dt}\right) + \left(\frac{dz}{du}\right), \quad \left(\frac{dz}{dy}\right) = \left(\frac{dz}{du}\right)$$

atque

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddz}{dt^2}\right) + 2\left(\frac{ddz}{dt\,du}\right) + \left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dx\,dy}\right) = \left(\frac{ddz}{dt\,du}\right) + \left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{du^2}\right).$$

COROLLARIUM 2

235. Etsi ergo hic est $t = x$, tamen non est $\left(\frac{dz}{dt}\right) = \left(\frac{dz}{dx}\right)$, cuius rei ratio est, quod in forma $\left(\frac{dz}{dx}\right)$ quantitas y sumitur constans, in $\left(\frac{dz}{dt}\right)$ vero quantitas $u = x + y$; id quod in genere notasse iuvat, ne ex aequalitate $t = x$ ad aequalitatem formularum $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dt}\right)$ concludamus.

EXEMPLUM 2

236. *Si inter variabiles t, u et x, y haec relatio constituatur, ut sit $t = \alpha x^m$ et $u = \beta y^n$, reductionem exhibere.*

Hic ergo erit

$$\left(\frac{dt}{dx}\right)=m\alpha x^{m-1},\quad \left(\frac{dt}{dy}\right)=0,\quad \left(\frac{ddt}{dx^2}\right)=m(m-1)\alpha x^{m-2},$$

$$\left(\frac{du}{dx}\right)=0,\quad \left(\frac{du}{dy}\right)=n\beta y^{n-1},\quad \left(\frac{ddu}{dy^2}\right)=n(n-1)\beta y^{n-2},$$

unde obtinemus pro formulis primi gradus

$$\left(\frac{dz}{dx}\right)=m\alpha x^{m-1}\left(\frac{dz}{dt}\right),\quad \left(\frac{dz}{dy}\right)=n\beta y^{n-1}\left(\frac{dz}{du}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{ddz}{dx^2}\right)=m(m-1)\alpha x^{m-2}\left(\frac{dz}{dt}\right)+mm\alpha\alpha x^{2m-2}\left(\frac{ddz}{dt^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right)=mn\alpha\beta x^{m-1}y^{n-1}\left(\frac{ddz}{dtdu}\right),$$

$$\left(\frac{ddz}{dy^2}\right)=n(n-1)\beta y^{n-2}\left(\frac{dz}{du}\right)+nn\beta\beta y^{2n-2}\left(\frac{ddz}{du^2}\right),$$

in quibus formulis iam loco x et y earum valores per t et u induci debent.

EXEMPLUM 3

237. *Si inter variabiles t, u et x, y haec relatio constituatur, ut sit $x=t$ et $\frac{x}{y}=u$, formularum differentialium reductionem exhibere.*

Cum sit $t=x$ et $u=\frac{x}{y}$, erit

$$\left(\frac{dt}{dx}\right)=1,\quad \left(\frac{dt}{dy}\right)=0$$

hincque formulae involventes ddt evanescunt. Porro

$$\left(\frac{du}{dx}\right)=\frac{1}{y}=\frac{u}{t},\quad \left(\frac{du}{dy}\right)=\frac{-x}{yy}=\frac{-uu}{t},$$

$$\left(\frac{ddu}{dx^2}\right)=0,\quad \left(\frac{ddu}{dxdy}\right)=\frac{-1}{yy}=\frac{-uu}{tt},\quad \left(\frac{ddu}{dy^2}\right)=\frac{2x}{y^3}=\frac{2u^3}{tt},$$

unde pro formulis primi gradus habebimus

$$\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dt}\right)+\frac{u}{t}\left(\frac{dz}{du}\right),\quad \left(\frac{dz}{dy}\right)=\frac{-uu}{t}\left(\frac{dz}{du}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddz}{dt^2}\right)+\frac{2u}{t}\left(\frac{ddz}{dtdu}\right)+\frac{uu}{tt}\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right)=\frac{-uu}{tt}\left(\frac{dz}{du}\right)-\frac{uu}{t}\left(\frac{ddz}{dtdu}\right)-\frac{u^3}{tt}\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right)=\frac{2u^3}{tt}\left(\frac{dz}{du}\right)+\frac{u^4}{tt}\left(\frac{ddz}{du^2}\right).$$

EXEMPLUM 4

238. *Si inter variabiles t, u et x, y haec relatio constituatur, ut sit $t=e^x$ et $u=e^x y$ seu $x=lt$ et $y=\frac{u}{t}$, reductionem formularum differentialium exhibere.*

Hic ergo est

$$\left(\frac{dt}{dx}\right)=e^x=t,\quad \left(\frac{dt}{dy}\right)=0,\quad \left(\frac{ddt}{dx^2}\right)=e^x=t,\quad \left(\frac{ddt}{dxdy}\right)=0.$$

Deinde

$$\left(\frac{du}{dx}\right)=e^x y=u,\quad \left(\frac{du}{dy}\right)=e^x=t,$$

tum vero

$$\left(\frac{ddu}{dx^2}\right)=e^x y=u,\quad \left(\frac{ddu}{dxdy}\right)=e^x=t,\quad \left(\frac{ddu}{dy^2}\right)=0.$$

Quare pro formulis primi gradus habebimus

$$\left(\frac{dz}{dx}\right)=t\left(\frac{dz}{dt}\right)+u\left(\frac{dz}{du}\right),\quad \left(\frac{dz}{dy}\right)=t\left(\frac{dz}{du}\right),$$

pro formulis autem secundi gradus

$$\left(\frac{ddz}{dx^2}\right)=t\left(\frac{dz}{dt}\right)+u\left(\frac{dz}{du}\right)+tt\left(\frac{ddz}{dt^2}\right)+2tu\left(\frac{ddz}{dtdu}\right)+uu\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right)=t\left(\frac{dz}{du}\right)+tt\left(\frac{ddz}{dtdu}\right)+tu\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right)=tt\left(\frac{ddz}{du^2}\right).$$

SCHOLION

239. In formulis generalibus § 232 datis assumsimus valores variabilium t et u per x et y expressos dari et universa evolutione facta tum demum pro x et y variabiles t et u restitui. Commodius ergo videatur, si statim variabilium x et y valores per t et u expressi habeantur; verum inde valores formularum $\left(\frac{dt}{dx}\right)$, $\left(\frac{dt}{dy}\right)$ etc. nimis complicate exprimerentur, quam ut eas in calculum introducere liceret. Scilicet si x et y per t et u dentur, inde fit

$$\left(\frac{dt}{dx}\right) = \frac{\left(\frac{dy}{du}\right)}{\left(\frac{dx}{dt}\right)\left(\frac{dy}{du}\right) - \left(\frac{dx}{du}\right)\left(\frac{dy}{dt}\right)}$$

ac formulae secundi gradus multo magis prodriturae sunt perplexae. Quovis ergo casu, quo huiusmodi reductione utendum videtur, coniectura potius quam certa ratione idoneam variabilium immutationem colligi conveniet.

Alia vero etiam datur reductio saepe insignem utilitatem afferens, dum ipsius functionis z quaesitae forma mutatur, veluti si ponatur $z = Vv$ denotante V functionem datam ipsarum x et y, ita ut iam v sit functio quaesita; quin etiam haec nova quaesita v alio modo cum datis implicari potest.

PROBLEMA 40

240. *Proposita functione z binarum variabilium x et y ac posito $z = Pv$, ita ut P sit data quaedam functio ipsarum x et y, formulas differentiales ipsius z per formulas differentiales novae functionis v exprimere.*

SOLUTIO

Cum sit $z = Pv$, ex regulis differentiandi traditis habebimus primo formulas differentiales primi gradus

$$\left(\frac{dz}{dx}\right) = \left(\frac{dP}{dx}\right)v + P\left(\frac{dv}{dx}\right) \quad \text{et} \quad \left(\frac{dz}{dy}\right) = \left(\frac{dP}{dy}\right)v + P\left(\frac{dv}{dy}\right).$$

Atque hinc deinceps formulae differentiales secundi ordinis ita prodibunt expressae

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddP}{dx^2}\right)v+2\left(\frac{dP}{dx}\right)\left(\frac{dv}{dx}\right)+P\left(\frac{ddv}{dx^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right)=\left(\frac{ddP}{dxdy}\right)v+\left(\frac{dP}{dx}\right)\left(\frac{dv}{dy}\right)+\left(\frac{dP}{dy}\right)\left(\frac{dv}{dx}\right)+P\left(\frac{ddv}{dxdy}\right),$$

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddP}{dy^2}\right)v+2\left(\frac{dP}{dy}\right)\left(\frac{dv}{dy}\right)+P\left(\frac{ddv}{dy^2}\right);$$

ubi cum P sit functio data ipsarum x et y, eius formulae differentiales simul habentur.

COROLLARIUM 1

241. Si P esset functio ipsius x tantum, puta X, tum posito $z=Xv$ erit

$$\left(\frac{dz}{dx}\right)=\frac{dX}{dx}v+X\left(\frac{dv}{dx}\right)\quad\text{et}\quad\left(\frac{dz}{dy}\right)=X\left(\frac{dv}{dy}\right),$$

tum

$$\left(\frac{ddz}{dx^2}\right)=\frac{ddX}{dx^2}v+\frac{2dX}{dx}\left(\frac{dv}{dx}\right)+X\left(\frac{ddv}{dx^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right)=\frac{dX}{dx}\left(\frac{dv}{dy}\right)+X\left(\frac{ddv}{dxdy}\right),$$

$$\left(\frac{ddz}{dy^2}\right)=X\left(\frac{ddv}{dy^2}\right).$$

COROLLARIUM 2

242. Transformatio haec easdem variabiles x et y servat et tantum loco functionis z alia v introducitur, cum ante manente eadem functione z binae variabiles x et y ad alias t et u sint reductae. Ex quo hae duae transformationes genere sunt diversae.

SCHOLION 1

243. Casus simplicior fuisset, si per additionem posuissemus $z=P+v$, ut esset P functio quaedam data ipsarum x et y; verum tum transformatio ita fit obvia, ut investigatione non indigeat; est enim manifesto

$$\left(\frac{dz}{dx}\right) = \left(\frac{dP}{dx}\right) + \left(\frac{dv}{dx}\right), \quad \left(\frac{dz}{dy}\right) = \left(\frac{dP}{dy}\right) + \left(\frac{dv}{dy}\right),$$

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddP}{dx^2}\right) + \left(\frac{ddv}{dx^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right) = \left(\frac{ddP}{dxdy}\right) + \left(\frac{ddv}{dxdy}\right),$$

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddP}{dy^2}\right) + \left(\frac{ddv}{dy^2}\right).$$

Neque vero etiam formas magis compositas evolvi necesse est, veluti si ponamus $z = \sqrt{(PP + vv)}$, quandoquidem talis forma vix unquam usum foret habitura.

SCHOLION 2

244. Praemissis his principiis et transformationibus negotium aggrediamur et methodos aperiamus ex data relatione inter formulas differentiales secundi gradus et primi gradus itemque ipsas quantitates principales harum ipsarum relationem investigandi. Hic scilicet praeter ipsas quantitates x, y et z earumque formulas differentiales primi gradus $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ considerandae veniunt tres formulae differentiales secundi gradus $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dxdy}\right)$ et $\left(\frac{ddz}{dy^2}\right)$, quarum vel una vel binae vel omnes tres in relationem propositam ingredi possunt, ubi insuper ingens discrimen formulae primi gradus, sive in relationem ingrediantur sive secus, constituunt. Non solum autem nimis longum foret omnes combinationes, uti in praecedente sectione fecimus, prosequi, sed etiam defectus idonearum methodorum impedit, quominus singula quaestionum huc pertinentium genera percurramus. Capita igitur pertractanda ita instituamus, prout methodus solvendi patietur, ea, ubi nihil praestare licet, penitus praetermissuri.

CAPUT II

DE UNA FORMULA DIFFERENTIALI SECUNDI GRADUS PER RELIQUAS QUANTITATES UTCUNQUE DATA

PROBLEMA 41

245. *Si z debeat esse eiusmodi functio ipsarum x et y, ut formula secundi gradus* $\left(\frac{ddz}{dx^2}\right)$ *aequetur functioni datae ipsarum x et y, indolem functionis z investigare.*

SOLUTIO

Sit P functio ista data ipsarum x et y, ita ut esse debeat

$$\left(\frac{ddz}{dx^2}\right) = P.$$

Sumatur iam y constans, et cum sit $d.\left(\frac{dz}{dx}\right) = dx\left(\frac{ddz}{dx^2}\right)$, erit

$$d.\left(\frac{dz}{dx}\right) = Pdx,$$

unde integrando prodit

$$\left(\frac{dz}{dx}\right) = \int Pdx + \text{Const.},$$

ubi in integratione $\int Pdx$ quantitas y pro constante habetur et constans adiicienda functionem quamcunque ipsius y denotabit, ita ut haec prima integratio praebeat

$$\left(\frac{dz}{dx}\right) = \int Pdx + f:y.$$

Nunc iterum quantitate y ut constante spectata erit $dz = dx\left(\frac{dz}{dx}\right)$ seu

$$dz = dx\int Pdx + dxf:y;$$

ubi cum $\int Pdx$ sit functio ipsarum x et y, quarum haec y constans assumitur, integratio denuo instituta dabit

$$z = \int dx\int Pdx + xf:y + F:y,$$

quod est integrale completum aequationis differentio-differentialis propositae $\left(\frac{ddz}{dx^2}\right) = P$, propterea quod duas functiones arbitrarias $f:y$ et $F:y$ complectitur, quarum utramque ita pro lubitu accipere licet, ut etiam functiones discontinuae non excludantur.

COROLLARIUM 1

246. Quodsi ergo proponatur haec conditio $\left(\frac{ddz}{dx^2}\right) = 0$, eius integratio completa dabit

$$z = xf:y + F:y$$

ob $P = 0$, cuius veritas ex differentiatione perspicitur, unde fit primo $\left(\frac{dz}{dx}\right) = f:y$, tum vero $\left(\frac{ddz}{dx^2}\right) = 0$.

COROLLARIUM 2

247. Eodem modo in genere integrale inventum per differentiationem comprobatur. Cum enim invenerimus

$$z = \int dx\int Pdx + xf:y + F:y,$$

prima differentiatio praebet

$$\left(\frac{dz}{dx}\right) = \int Pdx + f:y,$$

repetita vero $\left(\frac{ddz}{dx^2}\right) = P$.

COROLLARIUM 3

248. Simili modo si haec proponatur conditio $\left(\frac{ddz}{dy^2}\right) = Q$ existente Q functione quacunque ipsarum x et y, integrale completum reperitur

$$z = \int dy\int Qdy + yf:x + F:x,$$

ubi in geminato integrali $\int dy\int Qdy$ quantitas x pro constante habetur.

SCHOLION

249. Hinc ratio integralium completorum, quae ex formulis differentialibus secundi gradus nascuntur, in genere perspicitur, quae in hoc est sita, ut duae functiones arbitrariae invehantur; ubi iterum notandum est has functiones tam discontinuas quam continuas esse posse. Nisi ergo per totam hanc sectionem integralia duas huiusmodi functiones arbitrarias involvant, ea pro completis haberi nequeunt. Quotiescunque enim problema ad huiusmodi aequationem $\left(\frac{ddz}{dx^2}\right)=P$ perducit, eius indoles semper ita est comparata, ut tributo ipsi x certo quodam valore $x=a$ tam formula $\left(\frac{dz}{dx}\right)$ quam ipsa quantitas z datae cuipiam functioni ipsius y aequari possit. Quare si tam integrale $\int Pdx$ quam hoc $\int dx\int Pdx$ ita accipiatur, ut posito $x=a$ evanescat, erit pro eodem casu $x=a$ valor

$$\left(\frac{dz}{dx}\right)=f:y \quad \text{et} \quad z=af:y+F:y,$$

unde ex problematis natura utraque functio $f:y$ et $F:y$ definitur. Haec autem applicatio ad omnes casus fieri non posset, nisi integrale completum haberetur; quamobrem in hoc praecipue est incumbendum, ut omnium huiusmodi problematum integralia completa habeantur.

Ceterum hic in perpetuum monendum duco, quoties huiusmodi formula integralis $\int Pdx$ occurrit, semper solam quantitatem x variabilem accipi esse intelligendam, siquidem, si etiam y variabilis acciperetur, formula $\int Pdx$ ne significatum quidem admitteret. Simili modo in formula $\int dx\int Pdx$ intelligi debet in utraque integratione solam x variabilem assumi. Sin autem talis forma $\int dy\int Pdx$ occurrat, intelligendum est integrale $\int Pdx$ ex variabilitate solius x colligi debere; quod si ponatur $=R$, ut habeatur $\int Rdy$, hic iam sola y pro variabili erit habenda.

EXEMPLUM 1

250. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right)=\frac{xy}{a}.$$

Cum hic sit $P=\frac{xy}{a}$, erit

$$\int Pdx=\frac{xxy}{2a} \quad \text{et} \quad \int dx\int Pdx=\frac{x^3y}{6a}$$

sicque habebitur ex prima integratione

$$\left(\frac{dz}{dx}\right)=\frac{xxy}{2a}+f:y,$$

ita ut posito $x=a$ formula $\left(\frac{dz}{dx}\right)$ functioni cuicunque ipsius y aequari possit seu applicatae curvae cuiuscunque respondenti abscissae y. Tum vero altera integratione instituta erit

$$z=\frac{x^3y}{6a}+xf:y+F:y,$$

qui valor casu $x=a$ denuo functioni cuicunque ipsius y aequari potest.

EXEMPLUM 2

251. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right)=\frac{ax}{\sqrt{(xx+yy)}}.$$

Ob $P=\frac{ax}{\sqrt{(xx+yy)}}$ erit

$$\int Pdx=a\sqrt{(xx+yy)}$$

et

$$\int dx\int Pdx=a\int dx\sqrt{(xx+yy)}=\tfrac{1}{2}ax\sqrt{(xx+yy)}+\tfrac{1}{2}ayyl\left(x+\sqrt{(xx+yy)}\right),$$

unde prima integratio praebet

$$\left(\frac{dz}{dx}\right)=a\sqrt{(xx+yy)}+f:y,$$

altera vero

$$z=\tfrac{1}{2}ax\sqrt{(xx+yy)}+\tfrac{1}{2}ayyl\left(x+\sqrt{(xx+yy)}\right)+xf:y+F:y.$$

EXEMPLUM 3

252. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right)=\frac{1}{\sqrt{(aa-xx-yy)}}.$$

Cum sit $P = \frac{1}{\sqrt{(aa - xx - yy)}}$, erit

$$\int Pdx = \text{Ang. sin.} \frac{x}{\sqrt{(aa - yy)}},$$

tum vero

$$\int dx \int Pdx = x \text{ Ang. sin.} \frac{x}{\sqrt{(aa - yy)}} - \int \frac{xdx}{\sqrt{(aa - xx - yy)}}.$$

Quare integratio prima praebet

$$\left(\frac{dz}{dx}\right) = \text{Ang. sin.} \frac{x}{\sqrt{(aa - yy)}} + f : y$$

hincque ipsa functio quaesita erit

$$z = x \text{ Ang. sin.} \frac{x}{\sqrt{(aa - yy)}} + \sqrt{(aa - xx - yy)} + xf : y + F : y.$$

EXEMPLUM 4

253. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right) = x \sin. (x + y).$$

Ob $P = x \sin. (x + y)$ erit

$$\int Pdx = \int xdx \sin. (x + y) = -x \cos. (x + y) + \int dx \cos. (x + y)$$
$$= -x \cos. (x + y) + \sin. (x + y).$$

Tum vero est

$$\int xdx \cos. (x + y) = x \sin. (x + y) + \cos. (x + y)$$

ideoque

$$\int dx \int Pdx = -2 \cos. (x + y) - x \sin. (x + y).$$

Quocirca ambo nostra integralia erunt

$$\left(\frac{dz}{dx}\right) = \sin. (x + y) - x \cos. (x + y) + f : y$$

et

$$z = -2 \cos. (x + y) - x \sin. (x + y) + xf : y + F : y.$$

PROBLEMA 42

254. *Si z debeat esse eiusmodi functio variabilium x et y, ut sit*

$$\left(\frac{ddz}{dx^2}\right) = P\left(\frac{dz}{dx}\right) + Q$$

existentibus P et Q functionibus quibusvis ipsarum x et y, indolem functionis z in genere investigare.

SOLUTIO

Ponamus hic $\left(\frac{dz}{dx}\right) = v$, ut sit $\left(\frac{ddz}{dx^2}\right) = \left(\frac{dv}{dx}\right)$; erit nostra aequatio integranda

$$\left(\frac{dv}{dx}\right) = Pv + Q.$$

Spectetur ergo sola x ut variabilis et ob $dv = dx\left(\frac{dv}{dx}\right)$ erit

$$dv = Pvdx + Qdx,$$

quae per $e^{-\int Pdx}$ multiplicata et integrata dat

$$e^{-\int Pdx}v = \int e^{-\int Pdx}Qdx + f:y$$

ideoque

$$\left(\frac{dz}{dx}\right) = e^{\int Pdx}\int e^{-\int Pdx}Qdx + e^{\int Pdx}f:y.$$

Retineatur sola x variabilis spectata y ut constante et ob $dz = dx\left(\frac{dz}{dx}\right)$ erit

$$z = \int e^{\int Pdx}dx\int e^{-\int Pdx}Qdx + f:y\int e^{\int Pdx}dx + F:y,$$

quod ob binas functiones arbitrarias $f:y$ et $F:y$ est integrale completum.

COROLLARIUM 1

255. Problema hoc multo latius patet praecedente, cum conditio proposita etiam formulam primi gradus $\left(\frac{dz}{dx}\right)$ involvat; nihilo vero minus solutio feliciter successit.

COROLLARIUM 2

256. Hic ergo quadruplici integratione est opus. Primo scilicet quaeri debet integrale $\int P dx$; quod si ponatur $= lR$, quaeri porro debet integrale

$$\int e^{\int P dx} dx = \int R dx;$$

quod si ponamus $= S$, restat integrale

$$\int R dx \int \frac{Q dx}{R} = \int dS \int \frac{Q dx}{R},$$

quod abit in

$$S \int \frac{Q dx}{R} - \int \frac{QS dx}{R},$$

ita ut insuper hae duae formae integrari debeant.

COROLLARIUM 3

257. Eodem omnino modo resolvitur problema, quo esse debet

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{dz}{dy}\right) + Q,$$

si P et Q fuerint functiones quaecunque datae ipsarum x et y. Reperitur enim

$$\left(\frac{dz}{dy}\right) = e^{\int P dy} \int e^{-\int P dy} Q dy + e^{\int P dy} f : x$$

et

$$z = \int e^{\int P dy} dy \int e^{-\int P dy} Q dy + f : x \int e^{\int P dy} dy + F : x.$$

EXEMPLUM 1

258. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right) = \frac{n}{x}\left(\frac{dz}{dx}\right).$$

Posito $\left(\frac{dz}{dx}\right) = v$ sumtaque sola x variabili erit $\frac{dv}{dx} = \frac{nv}{x}$ ideoque $\frac{dv}{v} = \frac{n dx}{x}$, cuius integrale dat

$$v = \left(\frac{dz}{dx}\right) = x^n f : y.$$

Iam iterum sola x pro variabili habita erit

$$dz = x^n dx f : y,$$

cuius integrale completum est

$$z = \frac{1}{n+1} x^{n+1} f : y + F : y.$$

Casu autem $n = -1$ seu $\left(\frac{ddz}{dx^2}\right) = \frac{-1}{x}\left(\frac{dz}{dx}\right)$ erit

$$\left(\frac{dz}{dx}\right) = \frac{1}{x} f : y \quad \text{et} \quad z = lx \cdot f : y + F : y.$$

EXEMPLUM 2

259. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right) = \frac{n}{x}\left(\frac{dz}{dx}\right) + \frac{a}{xy}.$$

Posito $\left(\frac{dz}{dx}\right) = v$ sumtaque sola x variabili erit

$$dv = \frac{nv\,dx}{x} + \frac{a\,dx}{xy},$$

quae aequatio per x^n divisa et integrata praebet

$$\frac{v}{x^n} = \frac{a}{y}\int \frac{dx}{x^{n+1}} = \frac{-a}{nx^n y} + f : y$$

seu

$$v = \left(\frac{dz}{dx}\right) = \frac{-a}{ny} + x^n f : y.$$

Sit iterum sola x variabilis, ut habeatur

$$dz = \frac{-a\,dx}{ny} + x^n dx f : y,$$

prodibitque integrale completum

$$z = \frac{-ax}{ny} + \frac{1}{n+1} x^{n+1} f : y + F : y.$$

EXEMPLUM 3

260. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx^2}\right)=\frac{2nx}{xx+yy}\left(\frac{dz}{dx}\right)+\frac{x}{ay}.$$

Posito $\left(\frac{dz}{dx}\right)=v$ erit sumendo y constans

$$dv=\frac{2nxv\,dx}{xx+yy}+\frac{x\,dx}{ay},$$

quae aequatio per $(xx+yy)^n$ divisa et integrata dat

$$\frac{v}{(xx+yy)^n}=\frac{1}{ay}\int\frac{x\,dx}{(xx+yy)^n}=-\frac{1}{2(n-1)ay(xx+yy)^{n-1}}+f:y$$

seu

$$v=\left(\frac{dz}{dx}\right)=\frac{-(xx+yy)}{2(n-1)ay}+(xx+yy)^n f:y.$$

Hinc sumto iterum y constante fit

$$z=\frac{-x(xx+3yy)}{6(n-1)ay}+f:y\int(xx+yy)^n dx+F:y.$$

Casu, quo $n=1$ seu

$$\left(\frac{ddz}{dx^2}\right)=\frac{2x}{xx+yy}\left(\frac{dz}{dx}\right)+\frac{x}{ay},$$

erit

$$\frac{v}{xx+yy}=\frac{1}{ay}\int\frac{x\,dx}{xx+yy}=\frac{1}{2ay}l(xx+yy)+f:y,$$

hinc

$$\left(\frac{dz}{dx}\right)=\frac{xx+yy}{2ay}l(xx+yy)+(xx+yy)f:y$$

et

$$z=\frac{x(xx+3yy)}{6ay}l(xx+yy)-\frac{1}{9ay}\left(x^3+6xy^2-6y^3\,\text{Ang. tang.}\frac{x}{y}\right)$$
$$+\frac{1}{3}x(xx+3yy)f:y+F:y.$$

PROBLEMA 43

261. *Si z debeat esse eiusmodi functio binarum variabilium x et y, ut sit*

$$\left(\frac{ddz}{dx^2}\right) = P\left(\frac{dz}{dx}\right) + Q$$

existentibus P et Q functionibus quibuscunque datis omnium trium variabilium x, y et z, indolem functionis z investigare.

SOLUTIO

Posita quantitate y constante erit

$$\left(\frac{ddz}{dx^2}\right) = \frac{ddz}{dx^2} \quad \text{et} \quad \left(\frac{dz}{dx}\right) = \frac{dz}{dx}$$

ideoque habebitur aequatio differentialis secundi gradus ad librum praecedentem pertinens

$$ddz = Pdxdz + Qdx^2,$$

quae duas tantum variabiles x et z involvere est censenda, quia y in ea tanquam constans spectatur. Tentetur ergo integratio huius aequationis per methodos ibi expositas; quae si successerit, loco binarum constantium, quas duplex integratio invehit, scribantur ipsius y functiones indefinitae $f:y$ et $F:y$, quae adeo discontinuae accipi possunt, sicque habebitur aequationis propositae integrale completum.

COROLLARIUM 1

262. Reducitur ergo solutio huius problematis ad methodum integrandi in superiori libro traditam, ubi functionem unius variabilis ex data differentialium secundi gradus relatione investigari oportebat.

COROLLARIUM 2

263. Quodsi ergo resolutionem omnium aequationum differentialium secundi gradus, quae binas tantum variabiles involvunt, hic nobis concedi postulemus, solutio nostri problematis pro confecta est censenda.

COROLLARIUM 3

264. Me non monente intelligitur eodem modo aequationem

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{dz}{dy}\right) + Q$$

tractari oportere eiusque solutionem tanquam confectam spectari posse, quaecunque fuerint P et Q functiones ipsarum x, y et z.

SCHOLION 1

265. Ex solutionis ratione intelligitur problema multo latius patens simili modo resolvi posse. Si enim formula $\left(\frac{ddz}{dx^2}\right)$ quomodocunque per quantitates principales x, y, z ac praeterea formulam $\left(\frac{dz}{dx}\right)$ determinetur, ita ut etiam huius formulae $\left(\frac{dz}{dx}\right)$ potestates aliaeve functiones quaecunque ingrediantur, solutio semper ad librum superiorem revocabitur, quia ponendo y constans fit

$$\left(\frac{dz}{dx}\right) = \frac{dz}{dx} \quad \text{et} \quad \left(\frac{ddz}{dx^2}\right) = \frac{ddz}{dx^2}$$

ideoque resultat aequatio differentialis secundi gradus formae consuetae duas tantum variabiles x et z involvens. Hoc tantum teneatur loco constantium per utramque integrationem ingredientium scribi oportere formas $f:y$ et $F:y$. Satis igitur notabilem partem propositi nostri expedivimus, scilicet cum vel $\left(\frac{ddz}{dx^2}\right)$ utcunque per x, y, z et $\left(\frac{dz}{dx}\right)$, vel $\left(\frac{ddz}{dy^2}\right)$ utcunque per x, y, z et $\left(\frac{dz}{dy}\right)$ determinatur; ibi nempe excluditur formula primi gradus $\left(\frac{dz}{dy}\right)$, hic vero formula $\left(\frac{dz}{dx}\right)$. Quae si accederet, quaestio hac methodo neutiquam tractari posset, quemadmodum vel ex hoc casu simplicissimo $\left(\frac{ddz}{dx^2}\right) = \left(\frac{dz}{dy}\right)$ intelligere licet, cuius resolutio maxime ardua est putanda.

SCHOLION 2

266. Cum igitur trium formularum differentialium secundi gradus $\left(\frac{ddz}{dx^2}\right)$, $\left(\frac{ddz}{dx\,dy}\right)$, $\left(\frac{ddz}{dy^2}\right)$ primam ac tertiam hactenus sim contemplatus, quatenus earum per reliquas quantitates determinatio resolutionem admittit methodo quidem hic adhibita, superest, ut formulam quoque secundam $\left(\frac{ddz}{dx\,dy}\right)$ consideremus et, quibusnam determinationibus per reliquas quantitates x, y, z, $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ solutio absolvi queat, investigemus, in quo negotio a casibus simplicissimis exordiri conveniet.

PROBLEMA 44

267. *Si z eiusmodi debeat esse functio binarum variabilium x et y, ut fiat $\left(\frac{ddz}{dxdy}\right) = P$ existente P functione quacunque data ipsarum x et y, indolem functionis z generaliter determinare.*

SOLUTIO

Ponatur $\left(\frac{dz}{dx}\right) = v$ eritque $\left(\frac{ddz}{dxdy}\right) = \left(\frac{dv}{dy}\right)$ ideoque habebitur $\left(\frac{dv}{dy}\right) = P$. Iam spectetur quantitas x ut constans, ita ut P solam variabilem y contineat, eritque $dv = Pdy$, unde in hypothesi quantitatis x constantis integrando prodit

$$v = \left(\frac{dz}{dx}\right) = \int Pdy + f' : x,$$

ubi $\int Pdy$ erit functio data ipsarum x et y. Nunc porro spectetur x ut variabilis, y vero ut constans, ut adipiscamur hanc aequationem differentialem

$$dz = dx\int Pdy + dxf' : x,$$

quae integrata dat

$$z = \int dx \int Pdy + f : x + F : y;$$

ubi cum habeantur duae functiones arbitrariae, id indicio est hoc integrale esse completum.

COROLLARIUM 1

268. Si ordine inverso primum y, tum vero x constans posuissemus, invenissemus

$$\left(\frac{dz}{dy}\right) = \int Pdx + f' : y \quad \text{et} \quad z = \int dy \int Pdx + f : y + F : x,$$

qui valor aeque satisfacit ac praecedens.

COROLLARIUM 2

269. Patet ergo vel fore $\int dx \int Pdy = \int dy \int Pdx$ vel differentiam saltem exprimi per aggregatum ex functione ipsius x et functione ipsius y. Quod

etiam inde patet, quod posito

$$\int dx \int P dy = \int dy \int P dx = V$$

fiat utrinque $P = \left(\frac{ddV}{dx dy}\right)$.

COROLLARIUM 3

270. Si sit $P = 0$ seu debeat esse $\left(\frac{ddz}{dx dy}\right) = 0$, reperitur pro indole functionis z haec forma $z = f : x + F : y$.

SCHOLION

271. Hic casus in doctrina solidorum frequenter occurrit. Si enim natura superficiei exprimatur aequatione inter ternas coordinatas x, y et u, erit soliditas $= \int dx \int u dy$; quare si soliditas exprimatur per z, erit $\left(\frac{ddz}{dx dy}\right) = u$, ordinatae scilicet ad binas x et y normali. Tum vero si ponatur

$$du = p dx + q dy,$$

superficies huius solidi erit

$$= \int dx \int dy \sqrt{(1 + pp + qq)};$$

quae superficies si exprimatur littera z, erit

$$\left(\frac{ddz}{dx dy}\right) = \sqrt{(1 + pp + qq)}.$$

Quando ergo in nostro problemate eiusmodi functio z ipsarum x et y quaeritur, ut sit $\left(\frac{ddz}{dx dy}\right) = P$, idem est, ac si quaeratur soliditas respondens superficiei, cuius natura aequatione inter ternas coordinatas x, y et P exprimitur. Exemplis igitur aliquot hunc calculum illustremus.

EXEMPLUM 1

272. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx dy}\right) = \alpha x + \beta y.$$

Cum hic sit $P = \alpha x + \beta y$, erit

$$\int P dy = \alpha xy + \tfrac{1}{2}\beta yy \quad \text{et} \quad \int dx \int P dy = \tfrac{1}{2}\alpha xxy + \tfrac{1}{2}\beta xyy = \tfrac{1}{2}xy(\alpha x + \beta y),$$

unde functio quaesita z ita exprimitur, ut sit

$$z = \frac{1}{2}xy(\alpha x + \beta y) + f:x + F:y.$$

EXEMPLUM 2

273. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx\,dy}\right) = \sqrt{(aa - yy)}.$$

Hic est $P = \sqrt{(aa - yy)}$, ergo

$$\int P dx = x\sqrt{(aa - yy)},$$

ubi, quia perinde est, a variabilitate ipsius x incipio. Hinc igitur fit

$$\int dy \int P dx = x \int dy \sqrt{(aa - yy)} = \frac{1}{2}xy\sqrt{(aa - yy)} + \frac{1}{2}aax\int \frac{dy}{\sqrt{(aa - yy)}},$$

ex quo integrale completum erit

$$z = \frac{1}{2}xy\sqrt{(aa - yy)} + \frac{1}{2}aax \text{ Ang. sin. } \frac{y}{a} + f:x + F:y.$$

EXEMPLUM 3

274. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{ddz}{dx\,dy}\right) = \frac{a}{\sqrt{(aa - xx - yy)}}.$$

Ob $P = \dfrac{a}{\sqrt{(aa - xx - yy)}}$ erit

$$\int P dy = a \text{ Ang. sin. } \frac{y}{\sqrt{(aa - xx)}},$$

hinc

$$\int dx \int P dy = a \int dx \text{ Ang. sin. } \frac{y}{\sqrt{(aa - xx)}}.$$

Ponatur brevitatis gratia

$$\text{Ang. sin. } \frac{y}{\sqrt{(aa - xx)}} = \varphi;$$

erit

$$\int dx \int P dy = a \int \varphi dx = ax\varphi - a\int x dx \left(\frac{d\varphi}{dx}\right);$$

in hac enim integratione y pro constante habetur. Quare ob $\frac{y}{\sqrt{(aa-xx)}} = \sin.\varphi$ erit

$$\frac{yx}{(aa-xx)^{\frac{3}{2}}} = \left(\frac{d\varphi}{dx}\right)\cos.\varphi.$$

At vero est $\cos.\varphi = \frac{\sqrt{(aa-xx-yy)}}{\sqrt{(aa-xx)}}$ hincque

$$\left(\frac{d\varphi}{dx}\right) = \frac{yx}{(aa-xx)\sqrt{(aa-xx-yy)}}$$

et

$$\int xdx\left(\frac{d\varphi}{dx}\right) = y\int\frac{xxdx}{(aa-xx)\sqrt{(aa-xx-yy)}},$$

quo integrali invento erit

$$z = ax\,\text{Ang. sin.}\frac{y}{\sqrt{(aa-xx)}} - ay\int\frac{xxdx}{(aa-xx)\sqrt{(aa-xx-yy)}} + f:x + F:y,$$

quae forma per integrationem evoluta reducitur ad hanc

$$\begin{aligned} z = {} & ax\,\text{Ang. sin.}\frac{y}{\sqrt{(aa-xx)}} + ay\,\text{Ang. sin.}\frac{x}{\sqrt{(aa-yy)}} \\ & - aa\,\text{Ang. sin.}\frac{xy}{\sqrt{(aa-xx)(aa-yy)}} + f:x + F:y. \end{aligned}$$

Formulae enim

$$\int\frac{aadx}{(aa-xx)\sqrt{(aa-xx-yy)}}$$

integrale ita facillime elicitur. Ponatur

$$\frac{x}{\sqrt{(aa-xx-yy)}} = p;$$

erit $xx = \frac{pp(aa-yy)}{1+pp}$ et ob y constans per logarithmos differentiando

$$\frac{dx}{x} = \frac{dp}{p} - \frac{pdp}{1+pp} = \frac{dp}{p(1+pp)},$$

tum per illam formulam multiplicando

$$\frac{dx}{\sqrt{(aa-xx-yy)}} = \frac{dp}{1+pp}.$$

Porro est $aa - xx = \frac{aa + ppyy}{1 + pp}$, unde formula integralis fit

$$\int \frac{aadx}{(aa-xx)\sqrt{(aa-xx-yy)}} = \int \frac{aadp}{aa+ppyy} = \frac{aa}{yy}\int \frac{dp}{\frac{aa}{yy}+pp}$$

$$= \frac{a}{y}\,\text{Ang. tang.}\,\frac{py}{a} = \frac{a}{y}\,\text{Ang. tang.}\,\frac{xy}{a\sqrt{(aa-xx-yy)}}$$

$$= \frac{a}{y}\,\text{Ang. sin.}\,\frac{xy}{\sqrt{(aa-xx)(aa-yy)}}.$$

PROBLEMA 45

275. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut sit*

$$\left(\frac{ddz}{dx\,dy}\right) = P\left(\frac{dz}{dx}\right) + Q$$

existentibus P et Q functionibus quibuscunque ipsarum x et y, investigare indolem functionis z.

SOLUTIO

Ponatur $\left(\frac{dz}{dx}\right) = v$, ut oriatur ista aequatio $\left(\frac{dv}{dy}\right) = Pv + Q$, quae continet quantitates x, y et v; statuatur ergo x constans eritque

$$dv = Pvdy + Qdy,$$

quae per $e^{-\int Pdy}$ multiplicata praebet

$$e^{-\int Pdy}v = \int e^{-\int Pdy}Qdy + f' : x$$

ideoque

$$v = \left(\frac{dz}{dx}\right) = e^{\int Pdy}\int e^{-\int Pdy}Qdy + e^{\int Pdy}f' : x.$$

Nunc cum haec integralia determinate contineant x et y, spectetur y ut constans et sequens integratio praebet

$$z = \int e^{\int Pdy}dx \int e^{-\int Pdy}Qdy + \int e^{\int Pdy}dx f' : x + F : y,$$

quae integralia quovis casu evoluta fiunt manifesta.

COROLLARIUM 1

276. Ad hoc ergo problema resolvendum per integrationem primo quaeratur R, ut sit $\int Pdy = lR$; deinde quaeratur S, ut sit $\int \frac{Qdy}{R} = S$; denique sit $\int RSdx = T$, ita ut in illis sola quantitas y, hic vero sola x pro variabili habeatur. Quo facto erit nostrum integrale completum

$$z = T + \int Rdx f' : x + F : y.$$

COROLLARIUM 2

277. Hic ergo functio arbitraria $f : x$ in formula integrali est involuta; quae tamen si per applicatam curvae cuiuscunque respondentem abscissae x exhibeatur, hoc integrale $\int Rdx f' : x$ pro quovis valore ipsius y seorsim construi poterit, siquidem in hac integratione quantitas y ut constans spectatur.

SCHOLION

278. Eodem plane modo resolvitur permutandis variabilibus x et y hoc problema, quo functio z quaeritur, ut sit

$$\left(\frac{ddz}{dxdy}\right) = P\left(\frac{dz}{dy}\right) + Q,$$

dummodo P et Q sint functiones ipsarum x et y tantum ipsam functionem z non implicantes; solutio enim ita se habebit

$$z = \int e^{\int Pdx} dy \int e^{-\int Pdx} Qdx + \int e^{\int Pdx} dy f : y + F : x.$$

Quin etiam utrumque problema latius extendi potest, ac prius resolutionem admittet, si formula $\left(\frac{ddz}{dxdy}\right)$ aequetur functioni cuicunque trium quantitatum x, y et $\left(\frac{dz}{dx}\right)$, posterius vero, si $\left(\frac{ddz}{dxdy}\right)$ aequetur functioni cuicunque harum trium quantitatum x, y et $\left(\frac{dz}{dy}\right)$; utroque enim casu res reducitur ad aequationem differentialem primi gradus. Neque vero haec solvendi methodus succedit, si utraque formula primi gradus $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ simul ingrediatur vel si functiones P et Q etiam ipsam quantitatem z complectantur.

EXEMPLUM 1

279. *Quaeratur binarum variabilium x et y functio z, ut sit*

$$\left(\frac{ddz}{dxdy}\right)=\frac{n}{y}\left(\frac{dz}{dx}\right)+\frac{m}{x}.$$

Sit $\left(\frac{dz}{dx}\right)=v$; erit $\left(\frac{dv}{dy}\right)=\frac{nv}{y}+\frac{m}{x}$ et spectata x ut constante erit

$$dv=\frac{nvdy}{y}+\frac{mdy}{x},$$

unde per y^n dividendo prodit

$$\frac{v}{y^n}=\frac{m}{x}\int\frac{dy}{y^n}=\frac{-m}{(n-1)xy^{n-1}}+f':x,$$

ita ut sit

$$v=\left(\frac{dz}{dx}\right)=\frac{-my}{(n-1)x}+y^nf':x;$$

sumatur iam y constans et denuo integrando obtinetur

$$z=\frac{-m}{n-1}ylx+y^nf:x+F:y.$$

EXEMPLUM 2

280. *Quaeratur binarum variabilium x et y functio z, ut sit*

$$\left(\frac{ddz}{dxdy}\right)=\frac{y}{xx+yy}\left(\frac{dz}{dx}\right)+\frac{a}{xx+yy}.$$

Posito $\left(\frac{dz}{dx}\right)=v$ et sumto x constante erit

$$dv=\frac{vydy}{xx+yy}+\frac{ady}{xx+yy},$$

quae aequatio per $\sqrt{(xx+yy)}$ divisa dat

$$\frac{v}{\sqrt{(xx+yy)}}=a\int\frac{dy}{(xx+yy)^{\frac{3}{2}}}=\frac{ay}{xx\sqrt{(xx+yy)}}+f:x.$$

Ergo

$$v=\left(\frac{dz}{dx}\right)=\frac{ay}{xx}+\sqrt{(xx+yy)}\cdot f:x;$$

sit iam y constans reperieturque

$$z = \frac{-ay}{x} + \int f : x dx \sqrt{(xx + yy)} + F : y,$$

ubi quidem integrale

$$\int f : x dx \sqrt{(xx + yy)}$$

ob functionem indeterminatam $f : x$, etsi y constans ponitur, in genere exprimi nequit, ita ut explicate per y et functiones ipsius x exhiberi possit.

SCHOLION

281. Formula ergo secundi gradus $\left(\frac{ddz}{dxdy}\right)$ non tam largam casuum resolubilium copiam admittit quam binae reliquae $\left(\frac{ddz}{dx^2}\right)$ et $\left(\frac{ddz}{dy^2}\right)$, cum in his solutio succedat, etiamsi ipsa quantitas z quoque in earum determinationem ingrediatur, quod hic secus evenit, cum methodus non pateat huiusmodi aequationem $\left(\frac{ddz}{dxdy}\right) = P\left(\frac{dz}{dx}\right) + Q$, quando litterae P et Q quantitatem z continent, resolvendi; neque etiam solutio locum habet, quando praeter formulam primi gradus $\left(\frac{dz}{dx}\right)$ simul quoque altera adest. Interim tamen dantur casus, quibus solutiones particulares exhiberi possunt eaeque adeo infinitae, quae iunctim sumtae solutioni generali aequivalere videntur, etiamsi in applicatione ad usum practicum parum subsidii plerumque afferant; formas tamen huiusmodi solutionum notasse iuvabit.

PROBLEMA 46

282. *Si z eiusmodi debeat esse functio binarum variabilium x et y, ut fiat $\left(\frac{ddz}{dxdy}\right) = az$, indolem huius functionis z particulariter saltem investigare.*

SOLUTIO

Cum quantitas z unam ubique teneat dimensionem, evidens est, si statuatur $z = e^p q$, quantitatem exponentialem e^p ex calculo evanescere. Ponamus igitur $z = e^{\alpha x} Y$, ita ut Y functionem ipsius y tantum contineat, eritque

$$\left(\frac{dz}{dx}\right) = \alpha e^{\alpha x} Y \quad \text{et} \quad \left(\frac{ddz}{dxdy}\right) = \alpha e^{\alpha x} \frac{dY}{dy} = a e^{\alpha x} Y,$$

unde fit

$$\frac{\alpha dY}{Y} = a dy \quad \text{et} \quad Y = e^{\frac{ay}{\alpha}},$$

sicque iam solutionem particularem habemus

$$z = Ae^{\alpha x + \frac{ay}{\alpha}},$$

quae autem satis late patet, cum tam A quam α pro lubitu assumi possit. Plures autem valores ipsius z seorsim satisfacientes etiam iunctim sumti satisfaciunt, unde huiusmodi expressionem multo generaliorem deducimus

$$z = Ae^{\alpha x + \frac{a}{\alpha} y} + Be^{\beta x + \frac{a}{\beta} y} + Ce^{\gamma x + \frac{a}{\gamma} y} + De^{\delta x + \frac{a}{\delta} y} + \text{etc.};$$

ubi cum A, B, C etc., item α, β, γ etc. omnes valores possibiles recipere queant, haec forma pro maxime universali est habenda neque, si ad amplitudinem spectamus, quicquam cedere videtur superioribus solutionibus, quae binas functiones arbitrarias involvunt, propterea quod hic duplicis generis coefficientes arbitrarii occurrunt; interim tamen haud liquet, quomodo functiones discontinuae hac relatione repraesentari queant.

COROLLARIUM 1

283. Pro solutione ergo particulari invenienda sumantur bini numeri m et n, ut eorum productum sit $mn = a$, eritque $z = Ae^{mx+ny}$. Atque etiam ex iisdem numeris permutatis erit $z = Ae^{nx+my}$.

COROLLARIUM 2

284. Ex tali numerorum m et n pari, ut sit $mn = a$, solutiones quoque per sinus et cosinus exhiberi possunt; erit enim

$$z = B \sin. (mx - ny) \quad \text{vel} \quad z = B \cos. (mx - ny)$$

vel etiam permutando

$$z = B \sin. (nx - my) \quad \text{vel} \quad z = B \cos. (nx - my).$$

COROLLARIUM 3

285. Cum igitur huiusmodi formulae innumerabiles exhiberi queant, singulae per constantes quascunque multiplicatae et in unam summam collectae dabunt solutionem generalem problematis.

SCHOLION

286. Neque tamen haec solutio, etsi infinities infinitas determinationes recipit, ita est comparata, ut eiusmodi solutionibus, quae binas functiones arbitrarias involvunt, aequivalens aestimari possit; propterea quod non patet, quomodo singulas litteras assumi oporteat, ut pro dato casu, verbi gratia $y=0$, quantitas z vel $\left(\frac{dz}{dx}\right)$ seu $\left(\frac{dz}{dy}\right)$ datae functioni ipsius x aequalis evadat, cuiuscunque etiam indolis fuerit haec functio. Semper autem solutio generalis duplicis huiusmodi determinationis capax esse debet.

Quando autem talem solutionem impetrare non licet, utique eiusmodi solutionibus, uti hic invenimus, contenti esse debemus. Ac tales quidem solutiones simili modo obtinere possumus, si proponatur eiusmodi aequatio

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+Rz=0,$$

si modo litterae P, Q, R denotent functiones ipsius x tantum. Posito enim $z=e^{\alpha y}X$, ut X sit functio solius x, ob

$$\left(\frac{dz}{dx}\right)=e^{\alpha y}\frac{dX}{dx} \quad \text{et} \quad \left(\frac{dz}{dy}\right)=\alpha e^{\alpha y}X$$

erit

$$\frac{\alpha dX}{dx}+\frac{PdX}{dx}+\alpha QX+RX=0,$$

unde reperitur

$$\frac{dX}{X}=\frac{-dx(\alpha Q+R)}{\alpha+P},$$

sicque elicitur pro quovis numero α idoneus valor ipsius X. Quare sumendis infinitis numeris α hoc modo expressio infinities infinitas determinationes recipiens colligitur

$$z=Ae^{\alpha y}X+Be^{\beta y}X'+Ce^{\gamma y}X''+\text{etc.}$$

Verum tamen dantur etiam casus eiusmodi aequationum, quae solutiones vere completas admittunt, quarum rationem in sequente problemate indagemus.

PROBLEMA 47

287. *Proposita aequatione resolvenda*

$$\left(\frac{ddz}{dx\,dy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+Rz+S=0$$

investigare, cuiusmodi functiones ipsarum x et y esse debeant quantitates P, Q, R et S, ut haec aequatio solutionem vere completam admittat.

SOLUTIO

Sit V functio quaecunque ipsarum x et y ac ponatur $z=e^V v$, ita ut iam v sit quantitas incognita, cuius valorem investigari oporteat. Cum igitur sit

$$\left(\frac{dz}{dx}\right)=e^V\left(\left(\frac{dv}{dx}\right)+v\left(\frac{dV}{dx}\right)\right),\quad \left(\frac{dz}{dy}\right)=e^V\left(\left(\frac{dv}{dy}\right)+v\left(\frac{dV}{dy}\right)\right),$$

facta substitutione totaque aequatione per e^V divisa prodibit sequens aequatio

$$\left.\begin{aligned} e^{-V}S+\left(\frac{ddv}{dxdy}\right)+\left(\frac{dV}{dy}\right)\left(\frac{dv}{dx}\right)+\left(\frac{dV}{dx}\right)\left(\frac{dv}{dy}\right)&+\left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right)v\\ +\quad P\left(\frac{dv}{dx}\right)+\quad Q\left(\frac{dv}{dy}\right)&+\quad\left(\frac{ddV}{dxdy}\right)v\\ &+\quad P\left(\frac{dV}{dx}\right)v\\ &+\quad Q\left(\frac{dV}{dy}\right)v\\ &+\quad Rv \end{aligned}\right\}=0.$$

Efficiendum iam est, ut haec aequatio resolutionem completam admittat; cum igitur ante viderimus talem aequationem

$$\left(\frac{ddv}{dxdy}\right)+T\left(\frac{dv}{dx}\right)+e^{-V}S=0$$

generaliter resolvi posse, qualescunque etiam functiones ipsarum x et y pro [T], S et V accipiantur, ad hanc aequationem illam redigamus. Necesse igitur est statui

$$P + \left(\frac{dV}{dy}\right) = T, \quad Q + \left(\frac{dV}{dx}\right) = 0$$

et

$$R + Q\left(\frac{dV}{dy}\right) + P\left(\frac{dV}{dx}\right) + \left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right) + \left(\frac{ddV}{dx\,dy}\right) = 0,$$

unde obtinemus

$$P = T - \left(\frac{dV}{dy}\right), \quad Q = -\left(\frac{dV}{dx}\right) \quad \text{et} \quad R = \left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right) - T\left(\frac{dV}{dx}\right) - \left(\frac{ddV}{dx\,dy}\right).$$

Cum igitur per § 275 reperiatur

$$v = -\int e^{-\int T dy} dx \int e^{\int T dy - V} S dy + \int e^{-\int T dy} dx f : x + F : y,$$

erit aequationis propositae

$$\left(\frac{ddz}{dx\,dy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + Rz + S = 0,$$

si modo litterae P, Q, R assignatos teneant valores, integrale completum

$$z = -e^V \int e^{-\int T dy} dx \int e^{\int T dy - V} S dy + e^V \int e^{-\int T dy} dx f : x + e^V F : y,$$

quandoquidem hic formae $f : x$ et $F : y$ functiones quascunque ipsius x et ipsius y denotant.

COROLLARIUM 1

288. Quaecunque ergo functiones ipsarum x et y pro litteris T et V accipiantur, inde oriuntur valores idonei pro litteris P, Q, R assumendi, ut aequatio resolutionem completam admittat; functio autem S arbitrio nostro relinquitur.

COROLLARIUM 2

289. Possunt etiam in aequatione proposita functiones P et Q indefinitae relinqui eritque tum

$$V = -\int Q dx \quad \text{et} \quad \left(\frac{dV}{dy}\right) = -\int dx \left(\frac{dQ}{dy}\right) \quad \text{atque} \quad \left(\frac{ddV}{dx\,dy}\right) = -\left(\frac{dQ}{dy}\right),$$

unde tantum quantitas R ita determinari debet, ut sit

$$R - PQ - \left(\frac{dQ}{dy}\right) = 0 \quad \text{seu} \quad R = PQ + \left(\frac{dQ}{dy}\right).$$

COROLLARIUM 3

290. Quia hic pro $\int Qdx$ scribi potest $\int Qdx + Y$ denotante Y functionem quamcunque ipsius y, ob $V = -\int Qdx - Y$ complete integrabilis erit haec aequatio

$$\left(\frac{ddz}{dxdy}\right) + P\left(\frac{dz}{dx}\right) + Q\left(\frac{dz}{dy}\right) + \left(PQ + \left(\frac{dQ}{dy}\right)\right)z + S = 0,$$

cuius integrale est

$$z = e^{-\int Qdx - Y}v$$

existente

$$\left(\frac{ddv}{dxdy}\right) + \left(P - \int dx\left(\frac{dQ}{dy}\right) - \frac{dY}{dy}\right)\left(\frac{dv}{dx}\right) + e^{-V}S = 0$$

existente

$$T = P - \int dx\left(\frac{dQ}{dy}\right) - \frac{dY}{dy}$$

ac propterea

$$\int Tdy = \int Pdy - \int Qdx - Y,$$

unde valor ipsius v facile definitur.

SCHOLION

291. In hoc calculo, quo differentialia formularum integralium capi oportet, dum alia quantitas variabilis assumitur atque in integratione supponitur, haec regula est tenenda, quodsi fuerit $V = \int Qdx$, fore $\left(\frac{dV}{dy}\right) = \int dx\left(\frac{dQ}{dy}\right)$. Cum enim sit $\left(\frac{dV}{dx}\right) = Q$, erit $\left(\frac{ddV}{dxdy}\right) = \left(\frac{dQ}{dy}\right)$. Quodsi ergo statuatur $\left(\frac{dV}{dy}\right) = S$, erit $\left(\frac{dS}{dx}\right) = \left(\frac{dQ}{dy}\right)$ et $S = \left(\frac{dV}{dy}\right) = \int dx\left(\frac{dQ}{dy}\right)$; unde vicissim colligitur, si fuerit $S = \int dx\left(\frac{dQ}{dy}\right)$, fore ob $\int Sdy = V$ integrando $\int Sdy = \int Qdx$; quod cum ex principiis ante [§ 50 et 51] stabilitis per se sit manifestum, non opus esse iudico pro hoc quasi novo algorithmi genere praecepta seorsim tradere.

Videamus autem in aliquot exemplis, cuiusmodi aequationes ope huius methodi complete resolvere liceat.

EXEMPLUM 1

292. *Proposita aequatione differentio-differentiali*

$$\left(\frac{ddz}{dxdy}\right) + a\left(\frac{dz}{dx}\right) + b\left(\frac{dz}{dy}\right) + Rz + S = 0$$

definire indolem functionis R, ut haec aequatio resolutionem admittat existente S functione quacunque ipsarum x et y.

Cum sit $P = a$ et $Q = b$, erit $R = ab$ et $V = -bx$; tuto enim functio Y omitti potest, quia in sequente integratione iam binae functiones arbitrariae introducuntur; erit $T = a$. Unde posito $z = e^{-bx}v$ habebitur haec aequatio

$$\left(\frac{ddv}{dxdy}\right) + a\left(\frac{dv}{dx}\right) + e^{bx}S = 0$$

ac posito $\left(\frac{dv}{dx}\right) = u$ fit

$$\left(\frac{du}{dy}\right) + au + e^{bx}S = 0$$

et sumto x constante

$$e^{ay}u = -\int e^{ay+bx}Sdy + f' : x,$$

ergo

$$u = \left(\frac{dv}{dx}\right) = -e^{-ay}\int e^{ay+bx}Sdy + e^{-ay}f' : x$$

et sumto iam y constante

$$v = -e^{-ay}\int dx \int e^{ay+bx}Sdy + e^{-ay}f : x + F : y$$

sumendo $\int dxf' : x = f : x$. Quodsi iam pro $e^{-bx}f : x$ scribatur $f : x$, erit

$$z = -e^{-ay-bx}\int dx \int e^{ay+bx}Sdy + e^{-ay}f : x + e^{-bx}F : y.$$

ALITER

Si sumsissemus $V = -bx - ay$, prodiisset $T = a - a = 0$ ideoque posito $z = e^{-bx-ay}v$ quantitas v ex hac aequatione

$$\left(\frac{ddv}{dxdy}\right) + e^{bx+ay}S = 0$$

definiri deberet, quae dat

$$\left(\frac{dv}{dx}\right) = -\int e^{bx+ay}Sdy + f' : x \quad \text{et} \quad v = -\int dx \int e^{bx+ay}Sdy + f : x + F : y$$

et

$$z = e^{-bx-ay}\left(-\int dx \int e^{bx+ay}Sdy + f : x + F : y\right),$$

quae forma simplicior est praecedente, etiamsi eodem redeat, estque hoc integrale completum aequationis

$$\left(\frac{ddz}{dxdy}\right) + a\left(\frac{dz}{dx}\right) + b\left(\frac{dz}{dy}\right) + abz + S = 0.$$

EXEMPLUM 2

293. *Proposita aequatione differentio-differentiali*

$$\left(\frac{ddz}{dxdy}\right) + \frac{a}{y}\left(\frac{dz}{dx}\right) + \frac{b}{x}\left(\frac{dz}{dy}\right) + Rz + S = 0$$

definire indolem functionis R, ut haec aequatio resolutionem admittat existente S functione quacunque ipsarum x et y.

Cum sit $P = \frac{a}{y}$ et $Q = \frac{b}{x}$, erit $V = -blx - Y$ hincque $R = \frac{ab}{xy}$ et aequatio integrabilis erit

$$\left(\frac{ddz}{dxdy}\right) + \frac{a}{y}\left(\frac{dz}{dx}\right) + \frac{b}{x}\left(\frac{dz}{dy}\right) + \frac{ab}{xy}z + S = 0.$$

Quoniam igitur fit

$$T = P + \left(\frac{dV}{dy}\right) = \frac{a}{y} - \frac{dY}{dy},$$

sumamus $Y = +aly$, ut fiat $T = 0$, ac posito

$$z = e^{-blx-aly}v = x^{-b}y^{-a}v$$

quantitas v ex hac aequatione definiri debet

$$\left(\frac{ddv}{dxdy}\right) + x^b y^a S = 0,$$

unde fit

$$\left(\frac{dv}{dx}\right) = -x^b\int y^a S dy + f':x \quad \text{et} \quad v = -\int x^b dx \int y^a S dy + f:x + F:y$$

ideoque

$$z = \frac{-\int x^b dx \int y^a S dy + f:x + F:y}{x^b y^a}.$$

SCHOLION 1

294. Hinc igitur patet hanc aequationem ope istius methodi in genere integrari posse

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+\left(PQ+\left(\frac{dQ}{dy}\right)\right)z+S=0,$$

quaecunque functiones ipsarum x et y pro P, Q et S accipiantur. Ac resolutio quidem ita se habet, ut posito $z=e^{-\int Qdx-Y}v$ haec quantitas v determinetur hac aequatione

$$\left(\frac{ddv}{dxdy}\right)+\left(P-\int dx\left(\frac{dQ}{dy}\right)-\frac{dY}{dy}\right)\left(\frac{dv}{dx}\right)+e^{\int Qdx+Y}S=0,$$

ubi iam pro Y talis functio ipsius y accipi potest, ut huius aequationis forma simplicissima evadat, id quod potissimum evenit, si expressio

$$P-\int dx\left(\frac{dQ}{dy}\right)-\frac{dY}{dy}$$

ad nihilum redigi queat. In genere autem reperitur

$$v=-\int e^{-\int Pdy+\int Qdx+Y}dx\int e^{\int Pdy}Sdy+\int e^{-\int Pdy+\int Qdx+Y}dx\,f:x+F:y,$$

qui valor ergo per $e^{-\int Qdx-Y}$ multiplicatus praebet formam functionis z. Hoc modo autem functio Y ab arbitrio nostro pendens penitus e calculo egreditur fitque

$$z=-e^{-\int Qdx}\int e^{-\int Pdy+\int Qdx}dx\int e^{\int Pdy}Sdy+e^{-\int Qdx}\int e^{-\int Pdy+\int Qdx}dx\,f:x+e^{-\int Qdx}F:y,$$

quod est integrale completum huius aequationis

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+\left(PQ+\left(\frac{dQ}{dy}\right)\right)z+S=0.$$

SCHOLION 2

295. Permutandis autem variabilibus x et y etiam haec aequatio complete integrari potest

$$\left(\frac{ddz}{dxdy}\right)+P\left(\frac{dz}{dx}\right)+Q\left(\frac{dz}{dy}\right)+\left(PQ+\left(\frac{dP}{dx}\right)\right)z+S=0,$$

cuius integrale erit

$$z = -e^{-\int P dy}\int e^{-\int Q dx + \int P dy} dy \int e^{\int Q dx} S dx + e^{-\int P dy}\int e^{-\int Q dx + \int P dy} dy f : y + e^{-\int P dy} F : x,$$

ubi praecipue hic casus in utraque forma contentus notari meretur, si fuerit $P = Y$ et $Q = X$ existente X functione ipsius x et Y ipsius y tantum; tum enim huius aequationis

$$\left(\frac{ddz}{dxdy}\right) + Y\left(\frac{dz}{dx}\right) + X\left(\frac{dz}{dy}\right) + XYz + S = 0$$

integrale completum erit

$$z = -e^{-\int X dx - \int Y dy}\int e^{\int X dx} dx \int e^{\int Y dy} S dy + e^{-\int X dx - \int Y dy}(f : x + F : y),$$

quod etiam ita exhiberi potest

$$e^{\int X dx + \int Y dy} z = f : x + F : y - \int e^{\int X dx} dx \int e^{\int Y dy} S dy$$

vel etiam hoc modo

$$e^{\int X dx + \int Y dy} z = f : x + F : y - \int e^{\int Y dy} dy \int e^{\int X dx} S dx.$$

CAPUT III

SI DUAE VEL OMNES FORMULAE SECUNDI GRADUS PER RELIQUAS QUANTITATES DETERMINANTUR

PROBLEMA 48

296. *Si z eiusmodi debeat esse functio ipsarum x et y, ut fiat* $\left(\frac{ddz}{dy^2}\right) = aa\left(\frac{ddz}{dx^2}\right)$, *indolem functionis z determinare.*

SOLUTIO

Introducantur binae novae variabiles t et u, ut sit

$$t = \alpha x + \beta y \quad \text{et} \quad u = \gamma x + \delta y,$$

atque ex § 233 omnes formulae differentiales sequentes mutationes subibunt

$$\left(\frac{dz}{dx}\right) = \alpha\left(\frac{dz}{dt}\right) + \gamma\left(\frac{dz}{du}\right), \quad \left(\frac{dz}{dy}\right) = \beta\left(\frac{dz}{dt}\right) + \delta\left(\frac{dz}{du}\right),$$

$$\left(\frac{ddz}{dx^2}\right) = \alpha\alpha\left(\frac{ddz}{dt^2}\right) + 2\alpha\gamma\left(\frac{ddz}{dtdu}\right) + \gamma\gamma\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right) = \alpha\beta\left(\frac{ddz}{dt^2}\right) + (\alpha\delta + \beta\gamma)\left(\frac{ddz}{dtdu}\right) + \gamma\delta\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right) = \beta\beta\left(\frac{ddz}{dt^2}\right) + 2\beta\delta\left(\frac{ddz}{dtdu}\right) + \delta\delta\left(\frac{ddz}{du^2}\right),$$

unde nostra aequatio transibit in hanc

$$(\beta\beta - \alpha\alpha aa)\left(\frac{ddz}{dt^2}\right) + 2(\beta\delta - \alpha\gamma aa)\left(\frac{ddz}{dtdu}\right) + (\delta\delta - \gamma\gamma aa)\left(\frac{ddz}{du^2}\right) = 0.$$

Ponatur ergo

$$\beta\beta = \alpha\alpha aa \quad \text{et} \quad \delta\delta = \gamma\gamma aa$$

seu

$$\alpha = 1, \quad \gamma = 1, \quad \beta = a \quad \text{et} \quad \delta = -a,$$

ut binae formulae extremae evanescant, quod fit ponendo

$$t = x + ay \quad \text{et} \quad u = x - ay,$$

eritque

$$-2(aa + aa)\left(\frac{ddz}{dtdu}\right) = 0 \quad \text{seu} \quad \left(\frac{ddz}{dtdu}\right) = 0,$$

unde per § 270 colligitur integrale completum

$$z = f : t + F : u$$

ac pro t et u restitutis valoribus

$$z = f : (x + ay) + F : (x - ay),$$

quae forma manifesto satisfacit, cum sit

$$\left(\frac{dz}{dx}\right) = f' : (x + ay) + F' : (x - ay), \quad \left(\frac{dz}{dy}\right) = af' : (x + ay) - aF' : (x - ay),$$

$$\left(\frac{ddz}{dx^2}\right) = f'' : (x + ay) + F'' : (x - ay), \quad \left(\frac{ddz}{dy^2}\right) = aaf'' : (x + ay) + aaF'' : (x - ay).$$

COROLLARIUM 1

297. Valor igitur ipsius z aequatur aggregato duarum functionum arbitrariarum, alterius ipsius $x + ay$, alterius ipsius $x - ay$, atque ambae hae functiones ita ad arbitrium assumi possunt, ut etiam functiones discontinuas earum loco capere liceat.

COROLLARIUM 2

298. Pro lubitu ergo binae curvae quaecunque etiam libero manus tractu descriptae ad hunc usum adhiberi possunt. Scilicet si in una abscissa capiatur $= x + ay$, in altera vero abscissa $= x - ay$, summa applicatarum semper valorem idoneum pro functione z suppeditabit.

SCHOLION 1

299. Hoc fere primum est problema, quod in hoc novo calculi genere solvendum occurrit; perduxerat autem solutio generalis problematis de cordis vibrantibus ad hanc ipsam aequationem, quam hic tractavimus. Celeb. ALEMBERTUS, qui hoc problema primus felici successu est aggressus, methodo singulari aequationem integravit; scilicet cum esse oporteat $\left(\frac{ddz}{dy^2}\right) = a^2\left(\frac{ddz}{dx^2}\right)$, posito $dz = pdx + qdy$ indeque $dp = rdx + sdy$ et $dq = sdx + tdy$ illa aequatio postulat, ut sit $t = aar$. Consideratis porro istis aequationibus

$$dp = rdx + sdy, \quad dq = sdx + aardy$$

elicitur combinando

$$adp + dq = ar(dx + ady) + s(ady + dx)$$

seu

$$adp + dq = (ar + s)(dx + ady),$$

unde patet $ar + s$ functioni ipsius $x + ay$ aequari debere, ex quo etiam $ap + q$ tali functioni aequatur. Atque quia a aeque negative ac positive accipi potest, habentur duae huiusmodi aequationes

$$ap + q = 2af':(x + ay) \quad \text{et} \quad q - ap = 2aF':(x - ay),$$

unde colligitur

$$q = af':(x + ay) + aF':(x - ay) \quad \text{et} \quad p = f':(x + ay) - F':(x - ay),$$

hincque aequatio $dz = pdx + qdy$ sponte integratur fitque

$$z = f:(x + ay) - F:(x - ay).$$

Hoc modo sagacissimus Vir integrale completum est adeptus, sed non animadvertit loco functionum harum introductarum non solum omnis generis functiones continuas, sed etiam omni continuitatis lege destitutas accipi licere.[1])

1) ALEMBERTUS solutionem aequationis hic tractatae dedit pro casu $a = 1$ in Commentatione: *Recherches sur la courbe que forme une corde tenduë mise en vibration*, Mém. de l'acad. d. sc. de Berlin (1747), 1749, p. 214. Deinde EULERUS eandem solutionis formam evolvens animadvertit naturam problematis chordae vibrantis postulare, ut loco functionum arbitrariarum in solutionem ineuntium caperentur non solum functiones continuae, id est per operationes analyticas (§ 301)

SCHOLION 2

300. Cum plurimum intersit in hoc novo calculi genere quam plurimas methodos persequi, ab aliis solutio nostrae aequationis ita est tentata, ut ponerent $\left(\frac{dz}{dy}\right)=k\left(\frac{dz}{dx}\right)$, unde fit primo

$$\left(\frac{ddz}{dx\,dy}\right)=k\left(\frac{ddz}{dx^2}\right),$$

tum vero

$$\left(\frac{ddz}{dy^2}\right)=k\left(\frac{ddz}{dx\,dy}\right),$$

ex quo colligitur $\left(\frac{ddz}{dy^2}\right)=kk\left(\frac{ddz}{dx^2}\right)$.

Evidens ergo est pro nostro casu capi debere $kk=aa$ seu $k=\pm a$. Sit ergo $k=a$ et ob $\left(\frac{dz}{dy}\right)=a\left(\frac{dz}{dx}\right)$ fiet

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right)=\left(\frac{dz}{dx}\right)(dx+a\,dy).$$

hincque manifestum est fore $z=f:(x+ay)$ et ob a ambiguum, quoniam bini valores seorsim satisfacientes etiam iuncti satisfacient, concluditur ipsa solutio inventa.

Hoc etiam modo negotium confici potest. Statuatur

$$\left(\frac{ddz}{dy^2}\right)=aa\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddv}{dx\,dy}\right)$$

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conflatae, sed etiam discontinuae, scilicet tales, quae curvis libero manus ductu descriptis repraesentarentur (§ 37). Vide eius Commentationem 119 (indicis ENESTROEMIANI): *De vibratione chordarum exercitatio*, Nova acta erud. 1749, p. 512, cum versione Francogallica (Commentatio 140 indicis ENESTROEMIANI): *Sur la vibration des cordes*, Mém. de l'acad. d. sc. de Berlin (1748), 1750, p. 69; *LEONHARDI EULERI Opera omnia*, series II, vol. 8. Quod cum ALEMBERTUS non vellet concedere, sed assidue affirmaret functiones eas expressione analytica definiri debere, controversia orta est, quam EULERUS significat in § 37, p. 36 huius voluminis. Postea controversia illa renovata est, cum DANIEL BERNOULLI dedisset solutionem plane novam conflatam ex infinitis solutionibus particularibus per sinus et cosinus angulorum multiplorum expressis. Quaestiones hoc modo in medium prolatae saepe et diu agitatae sunt inter geometras XVIII. saeculi, sed earum resolutio saeculo demum sequente est inventa, scilicet in doctrina generali serierum trigonometricarum a FOURIER inaugurata. Cf. *LEONHARDI EULERI Opera omnia*, vol. citatum, et H. BURKHARDT, *Entwicklungen nach oscillirenden Functionen etc.*, Jahresber. d. Dtsch. Mathem. Ver. Bd. X, Heft 2, p. 10—47, Leipzig 1908. F. E.

eritque

$$\left(\frac{dz}{dy}\right) = \left(\frac{dv}{dx}\right) \quad \text{et} \quad aa\left(\frac{dz}{dx}\right) = \left(\frac{dv}{dy}\right).$$

Inventis nunc formulis primi gradus $\left(\frac{dv}{dx}\right)$ et $\left(\frac{dv}{dy}\right)$ ob

$$dv = dx\left(\frac{dv}{dx}\right) + dy\left(\frac{dv}{dy}\right)$$

habebimus has aequationes

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right) \quad \text{et} \quad dv = dx\left(\frac{dz}{dy}\right) + aady\left(\frac{dz}{dx}\right),$$

ex quarum combinatione colligimus

$$dv + adz = (dx + ady)\left(\left(\frac{dz}{dy}\right) + a\left(\frac{dz}{dx}\right)\right)$$

hincque

$$v + az = f:(x + ay) \quad \text{et} \quad v - az = F:(x - ay),$$

sicque pro z eadem forma exsurgit.

Methodus vero, quam in solutione sum secutus, ad naturam rei magis videtur accommodata, cum etiam in aliis problematibus magis complicatis insignem utilitatem afferat.

SCHOLION 3

301. Nostra autem solutio hoc habet incommodi, quod pro hac aequatione

$$\left(\frac{ddz}{dy^2}\right) + aa\left(\frac{ddz}{dx^2}\right) = 0$$

ad expressionem imaginariam deducit, scilicet

$$z = f:(x + ay\sqrt{-1}) + F:(x - ay\sqrt{-1});$$

quoties autem functiones f et F sunt continuae, cuiuscunque demum fuerint indolis, semper earum valores ad hanc formam $P \pm Q\sqrt{-1}$ reduci possunt, unde sequens forma ex illa facile deducenda semper valorem realem exhibebit

$$z = \frac{1}{2}f:(x + ay\sqrt{-1}) + \frac{1}{2}f:(x - ay\sqrt{-1})$$

$$+ \frac{1}{2\sqrt{-1}}F:(x + ay\sqrt{-1}) - \frac{1}{2\sqrt{-1}}F:(x - ay\sqrt{-1}),$$

pro cuius ad realitatem reductione notasse iuvabit posito

$$x = s\cos.\varphi \quad \text{et} \quad ay = s\sin.\varphi$$

fore

$$(x \pm ay\sqrt{-1})^n = s^n(\cos.n\varphi \pm \sqrt{-1}\cdot\sin.n\varphi).$$

Quare quoties functiones propositae per operationes analyticas sunt conflatae, hoc est continuae, earum valores realiter per cosinus et sinus angulorum multiplorum ipsius φ exhiberi possunt. Quando autem functiones illae sunt discontinuae, talis reductio neutiquam locum habet, etiamsi certum videatur etiam tunc formam allatam valorem realem esse adepturam. Quis autem in curva quacunque libero manus ductu descripta applicatas abscissis

$$x + ay\sqrt{-1} \quad \text{et} \quad x - ay\sqrt{-1}$$

respondentes animo saltem imaginari ac summam earum realem assignare valuerit aut differentiam, quae per $\sqrt{-1}$ divisa etiam erit realis? Hic ergo haud exiguus defectus calculi cernitur, quem nullo adhuc modo supplere licet; atque ob hunc ipsum defectum huiusmodi solutiones universales plurimum de sua vi perdunt.

PROBLEMA 49

302. *Proposita aequatione* $\left(\frac{ddz}{dy^2}\right) = PP\left(\frac{ddz}{dx^2}\right)$ *inquirere, quales functiones ipsarum x et y pro P assumere liceat, ut integratio ope reductionis succedat.*

SOLUTIO

Reductionem hanc ita fieri assumo, ut loco x et y binae aliae variabiles t et u introducantur, qua substitutione secundum § 232 in genere facta prodit haec aequatio

$$\left.\begin{matrix} +\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right) + \left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right) + \left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right) + 2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dt\,du}\right) + \left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right) \\ -P^2\left(\frac{ddt}{dx^2}\right) \qquad -P^2\left(\frac{ddu}{dx^2}\right) \qquad -P^2\left(\frac{dt}{dx}\right)^2 \qquad -2P^2\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right) \qquad -P^2\left(\frac{du}{dx}\right)^2 \end{matrix}\right\} = 0.$$

Iam relatio inter binas variabiles t, u et praecedentes x, y eiusmodi statuatur,

ut binae formulae $\left(\frac{ddz}{dt^2}\right)$ et $\left(\frac{ddz}{du^2}\right)$ ex calculo egrediantur, id quod fiet ponendo

$$\left(\frac{dt}{dy}\right)+P\left(\frac{dt}{dx}\right)=0 \quad \text{et} \quad \left(\frac{du}{dy}\right)-P\left(\frac{du}{dx}\right)=0.$$

Tum autem erit

$$\left(\frac{ddt}{dy^2}\right)=-P\left(\frac{ddt}{dx\,dy}\right)-\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right),$$

at cum sit indidem

$$\left(\frac{ddt}{dx\,dy}\right)=-P\left(\frac{ddt}{dx^2}\right)-\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right),$$

erit

$$\left(\frac{ddt}{dy^2}\right)=PP\left(\frac{ddt}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)-\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right)$$

similique modo sumendo P negative

$$\left(\frac{ddu}{dy^2}\right)=PP\left(\frac{ddu}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{du}{dx}\right)+\left(\frac{dP}{dy}\right)\left(\frac{du}{dx}\right).$$

Quibus substitutis nostra aequatio hanc induet formam

$$\left(P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)\right)\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)+\left(P\left(\frac{dP}{dx}\right)+\left(\frac{dP}{dy}\right)\right)\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right)$$
$$-4PP\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dt\,du}\right)=0;$$

quae cum unicam formulam secundi gradus $\left(\frac{ddz}{dt\,du}\right)$ contineat, integrationem admittit, si vel $\left(\frac{dz}{dt}\right)$ vel $\left(\frac{dz}{du}\right)$ e calculo excesserit. Ponamus ergo insuper

$$P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)=0,$$

qua aequatione indoles quaesitae functionis P definitur; quo facto aequatio integranda per $2P\left(\frac{du}{dx}\right)$ divisa erit

$$\left(\frac{dP}{dx}\right)\left(\frac{dz}{du}\right)-2P\left(\frac{dt}{dx}\right)\left(\frac{ddz}{dt\,du}\right)=0,$$

cuius integrale posito $\left(\frac{dz}{du}\right)=v$ fit

$$2lv=\int\frac{dt\left(\frac{dP}{dx}\right)}{P\left(\frac{dt}{dx}\right)}=2l\left(\frac{dz}{du}\right).$$

Verum prius ipsam functionem P per x et y definiri oportet.

Cum sit $\left(\frac{dP}{dy}\right) = P\left(\frac{dP}{dx}\right)$, erit

$$dP = dx\left(\frac{dP}{dx}\right) + Pdy\left(\frac{dP}{dx}\right)$$

hincque ponendo brevitatis ergo $\left(\frac{dP}{dx}\right) = p$ fit $dx = \frac{dP}{p} - Pdy$ atque

$$x = -Py + \int dP\left(y + \frac{1}{p}\right).$$

Statuatur ergo $y + \frac{1}{p} = f' : P$ ac reperitur

$$x + Py = f : P$$

et

$$p = \left(\frac{dP}{dx}\right) = \frac{1}{f' : P - y} \quad \text{ac} \quad \left(\frac{dP}{dy}\right) = \frac{P}{f' : P - y},$$

unde ratio determinationis quantitatis P per x et y definitur.

Pro novis autem variabilibus t et u ob $\left(\frac{dt}{dy}\right) = -P\left(\frac{dt}{dx}\right)$ erit

$$dt = \left(\frac{dt}{dx}\right)(dx - Pdy)$$

et ob $x = -Py + f : P$ fit

$$dt = \left(\frac{dt}{dx}\right)\left(dPf' : P - 2Pdy - ydP\right) = P^{\frac{1}{2}}\left(\frac{dt}{dx}\right)\left(\frac{dP}{\sqrt{P}}f' : P - 2dy\sqrt{P} - \frac{ydP}{\sqrt{P}}\right);$$

cuius postremae formulae cum integrale sit $\int \frac{dP}{\sqrt{P}} f' : P - 2y\sqrt{P}$, erit

$$t = F : \left(\int \frac{dP}{\sqrt{P}} f' : P - 2y\sqrt{P}\right).$$

Deinde ob $\left(\frac{du}{dy}\right) = P\left(\frac{du}{dx}\right)$ habetur

$$du = \left(\frac{du}{dx}\right)(dx + Pdy) = \left(\frac{du}{dx}\right)(dPf' : P - ydP)$$

ideoque

$$du = \left(\frac{du}{dx}\right)(f' : P - y)\,dP,$$

quare u aequabitur functioni ipsius P. In hoc autem negotio functiones

quascunque accipere licet, quia sequente demum integratione universalitas solutionis obtinetur. Quare ponamus

$$t = \int \frac{dP}{\sqrt{P}} f' : P - 2y\sqrt{P} \quad \text{et} \quad u = P$$

existente $x + Py = f : P$.

Denique ad ipsum integrale inveniendum, quia est

$$2l\left(\frac{dz}{du}\right) = \int \frac{dt\left(\frac{dP}{dx}\right)}{P\left(\frac{dt}{dx}\right)},$$

in qua integratione u seu P sumitur constans, per superiora est

$$\frac{dt}{\left(\frac{dt}{dx}\right)} = dPf' : P - 2Pdy - ydP = -2Pdy$$

ob P constans et $\left(\frac{dP}{dx}\right) = \frac{1}{f' : P - y}$, unde fit

$$2l\left(\frac{dz}{dP}\right) = \int \frac{-2dy}{f' : P - y} = 2l(f' : P - y) + 2lF : P$$

seu

$$\left(\frac{dz}{dP}\right) = (f' : P - y)F : P$$

hincque porro

$$z = \int dP(f' : P - y)F : P$$

sumendo hic t constans. Cum igitur sit

$$y = \frac{1}{2\sqrt{P}} \int \frac{dP}{\sqrt{P}} f' : P - \frac{t}{2\sqrt{P}},$$

ideoque

$$f' : P - y = f' : P - \frac{1}{2\sqrt{P}} \int \frac{dP}{\sqrt{P}} f' : P + \frac{t}{2\sqrt{P}},$$

unde conficitur

$$z = \int dP\left(f' : P - \frac{1}{2\sqrt{P}} \int \frac{dP}{\sqrt{P}} f' : P\right)F : P + \left(\frac{1}{2} \int \frac{dP}{\sqrt{P}} f' : P - y\sqrt{P}\right) \int \frac{dP}{\sqrt{P}} F : P$$
$$+ \Phi : \left(\int \frac{dP}{\sqrt{P}} f' : P - 2y\sqrt{P}\right),$$

quae expressio duas continet functiones arbitrarias F et Φ.

COROLLARIUM 1

303. Primum huius formae membrum ita transformari potest

$$\int\frac{dP}{\sqrt{P}}\Big(\sqrt{P}\cdot f':P-\tfrac{1}{2}\int\frac{dP}{\sqrt{P}}f':P\Big)F:P,$$

at

$$\sqrt{P}\cdot f':P-\tfrac{1}{2}\int\frac{dP}{\sqrt{P}}f':P=\int dP\sqrt{P}\cdot f'':P,$$

unde primum membrum erit

$$\int\frac{dP}{\sqrt{P}}F:P\int dP\sqrt{P}\cdot f'':P.$$

COROLLARIUM 2

304. Cum autem hoc primum membrum sit functio indefinita ipsius P, si ea indicetur per $\Pi:P$, erit

$$\frac{dP}{\sqrt{P}}F:P=\frac{dP\,\Pi':P}{\int dP\sqrt{P}\cdot f'':P},$$

unde forma integralis fit

$$z=\Pi:P+\Phi:\Big(\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\Big)+\Big(\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\Big)\int\frac{dP\,\Pi':P}{2\int dP\sqrt{P}\cdot f'':P}.$$

COROLLARIUM 3

305. Solutio magis particularis nascitur sumendo $\Pi:P=0$ hincque z aequabitur functioni cuicunque quantitatis $\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}$, quae ob $x+Py=f:P$ per x et y exhiberi censenda est.

SCHOLION

306. Quanquam hic eadem methodo sum usus atque in problemate praecedente [§ 296], tamen, quod mirum videatur, casus praecedentis problematis, quo erat $P=a$, in hac solutione non continetur. Ratio huius paradoxi in resolutione aequationis $\left(\frac{dP}{dy}\right)=P\left(\frac{dP}{dx}\right)$ est sita, cui manifesto satisfacit valor

$P=a$, etiamsi in forma inde derivata $x+Py=f:P$ non contineatur. Hic scilicet simile quiddam usu venit, quod iam supra[1]) observavimus, saepe aequationi differentiali valorem quendam satisfacere posse, qui in integrali non contineatur, veluti aequationi $dy\sqrt{(a-x)}=dx$ satisfacere videmus valorem $x=a$, quem tamen [forma] integralis $y=C-2\sqrt{(a-x)}$ excludit. Quare etiam nostro casu valor $P=a$ peculiarem evolutionem postulat in priore problemate peractam.

De reliquis, ubi pro $f:P$ certa quaedam functio ipsius P assumitur, exempla quaedam evolvamus.

EXEMPLUM 1

307. *Sumto* $f:P=0$, *ut sit* $P=-\frac{x}{y}$, *integrale completum huius aequationis*

$$\left(\frac{ddz}{dy^2}\right)=\frac{xx}{yy}\left(\frac{ddz}{dx^2}\right)$$

investigare.

Cum sit $f':P=0$, solutio inventa ob $\int\frac{dP}{\sqrt{P}}f':P=C$ praebet

$$z=\frac{-C}{2}\int\frac{dP}{\sqrt{P}}F:P+\left(\frac{1}{2}C-y\sqrt{P}\right)\int\frac{dP}{\sqrt{P}}F:P+\Phi:(C-2y\sqrt{P}).$$

Statuatur $\int\frac{dP}{\sqrt{P}}F:P=\Pi:P$ prodibitque

$$z=-y\sqrt{P}\cdot\Pi:P+\Phi:y\sqrt{P}.$$

Restituatur pro P valor $-\frac{x}{y}$ et ob $y\sqrt{P}=\sqrt{-xy}$ imaginarium $\sqrt{-1}$ in functiones involvendo erit

$$z=\sqrt{xy}\cdot\Pi:\frac{x}{y}+\Phi:\sqrt{xy},$$

quae forma facile in hanc transfunditur

$$z=x\Gamma:\frac{x}{y}+\Theta:xy,$$

ubi $x\Gamma:\frac{x}{y}$ denotat functionem quamcunque homogeneam unius dimensionis ipsarum x et y.

1) Vide *Institutionum calculi integralis* vol. I, § 546, *Leonhardi Euleri Opera omnia*, series I, vol. 11, p. 346. F. E.

Resolutio autem instituetur loco x et y has novas variabiles t et u introducendo, ut sit $t = C - 2\sqrt{-xy}$ et $u = -\frac{x}{y}$ vel etiam simplicius $t = 2\sqrt{xy}$ et $u = \frac{x}{y}$, unde fit

$$\left(\frac{dt}{dx}\right) = \frac{\sqrt{y}}{\sqrt{x}}, \quad \left(\frac{dt}{dy}\right) = \frac{\sqrt{x}}{\sqrt{y}}, \quad \left(\frac{ddt}{dx^2}\right) = \frac{-\sqrt{y}}{2x\sqrt{x}}, \quad \left(\frac{ddt}{dy^2}\right) = \frac{-\sqrt{x}}{2y\sqrt{y}},$$
$$\left(\frac{du}{dx}\right) = \frac{1}{y}, \quad \left(\frac{du}{dy}\right) = \frac{-x}{yy}, \quad \left(\frac{ddu}{dx^2}\right) = 0, \quad \left(\frac{ddu}{dy^2}\right) = \frac{2x}{y^3},$$

et ob $PP = \frac{xx}{yy}$ aequatio proposita hanc induit formam

$$0 \cdot \left(\frac{dz}{dt}\right) + \frac{2x}{y^3}\left(\frac{dz}{du}\right) - \frac{4x\sqrt{x}}{yy\sqrt{y}}\left(\frac{ddz}{dtdu}\right) = 0.$$

Nunc cum sit $ttu = 4xx$ et $x = \frac{1}{2}t\sqrt{u}$ atque $y = \frac{t}{2\sqrt{u}}$, habebimus

$$\frac{8uu}{tt}\left(\frac{dz}{du}\right) - \frac{8uu}{t}\left(\frac{ddz}{dtdu}\right) = 0 \quad \text{seu} \quad \left(\frac{dz}{du}\right) = t\left(\frac{ddz}{dtdu}\right).$$

Fiat $\left(\frac{dz}{du}\right) = v$, ut sit $v = t\left(\frac{dv}{dt}\right)$ et sumto u constante $\frac{dt}{t} = \frac{dv}{v}$, ergo $v = \left(\frac{dz}{du}\right) = tf' : u$. Sit iam t constans fietque

$$z = tf : u + F : t = 2\sqrt{xy} \cdot f : \frac{x}{y} + F : \sqrt{xy}$$

ut ante.

COROLLARIUM

308. Quemadmodum autem expressio inventa $z = x\Gamma : \frac{x}{y} + \Theta : xy$ satisfaciat, differentialibus rite sumtis perspicietur

$$\left(\frac{dz}{dx}\right) = \Gamma : \frac{x}{y} + \frac{x}{y}\Gamma' : \frac{x}{y} + y\Theta' : xy, \quad \left(\frac{dz}{dy}\right) = \frac{-xx}{yy}\Gamma' : \frac{x}{y} + x\Theta' : xy,$$

unde porro fit

$$\left(\frac{ddz}{dx^2}\right) = \frac{2}{y}\Gamma' : \frac{x}{y} + \frac{x}{yy}\Gamma'' : \frac{x}{y} + yy\Theta'' : xy$$

et

$$\left(\frac{ddz}{dy^2}\right) = \frac{2xx}{y^3}\Gamma' : \frac{x}{y} + \frac{x^3}{y^4}\Gamma'' : \frac{x}{y} + xx\Theta'' : xy.$$

EXEMPLUM 2

309. *Sumto* $f:P = \frac{PP}{2a}$, *ut sit*

$$PP = 2aPy + 2ax \quad et \quad P = ay + \sqrt{(aayy + 2ax)},$$

huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(2aayy + 2ax + 2ay\sqrt{(aayy + 2ax)}\right)\left(\frac{ddz}{dx^2}\right)$$

integrale completum investigare.

Cum sit $f:P = \frac{PP}{2a}$, erit $f':P = \frac{P}{a}$ et

$$\int \frac{dP}{\sqrt{P}} f':P = \int \frac{1}{a} dP\sqrt{P} = \frac{2}{3a} P\sqrt{P},$$

unde forma generalis supra [§ 302] inventa abit in

$$z = \int dP \frac{2P}{3a} F:P + \left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right)\int \frac{dP}{\sqrt{P}} F:P + \Phi:\left(\frac{2}{3a} P\sqrt{P} - 2y\sqrt{P}\right);$$

statuatur

$$\int \frac{dP}{\sqrt{P}} F:P = \Pi:P;$$

erit $dPF:P = dP\sqrt{P}\cdot\Pi':P$ atque

$$z = \frac{2}{3a}\int P^{\frac{3}{2}} dP\Pi':P + \left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right)\Pi:P + \Phi:\left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right).$$

Est autem

$$\frac{P}{3a} - y = \frac{-2}{3} y + \frac{1}{3}\sqrt{\left(yy + \frac{2x}{a}\right)};$$

quarum formularum evolutio deducit ad expressiones nimis perplexas. At substitutiones ad scopum perducentes sunt

$$t = \frac{2}{3a} P\sqrt{P} - 2y\sqrt{P} \quad \text{et} \quad u = P.$$

COROLLARIUM

310. Si pro solutione magis restricta ponatur $\Pi : P = P^{n-\frac{1}{2}}$, erit

$$\Pi' : P = \left(n - \frac{1}{2}\right) P^{n-\frac{3}{2}}$$

hincque colligitur

$$z = \frac{n}{(n+1)a} P^{n+1} - P^n y + \Phi : \left(\frac{P\sqrt{P}}{3a} - y\sqrt{P}\right).$$

Sit $n = 1$ et functio Φ evanescat; erit

$$z = \frac{1}{2a} PP - Py = x;$$

at casus $n = 2$ dat

$$z = \frac{2}{3a} P^3 - P^2 y = \frac{2}{3} axy + \frac{2}{3} P(2x + ayy)$$

seu

$$z = \frac{2}{3} aay^3 + 2axy + \frac{2}{3}(ayy + 2x)\sqrt{(aayy + 2ax)}.$$

SCHOLION

311. Forma integralis inventa [§ 302] sequenti modo simplicior effici potest. Ponatur $\int \frac{dP}{\sqrt{P}} F : P = \Pi : P$; erit

$$F : P = \sqrt{P} \cdot \Pi' : P$$

eritque omittendo postremum membrum

$$\Phi : \left(\int \frac{dP}{\sqrt{P}} f' : P - 2y\sqrt{P}\right),$$

quod nulla reductione indiget,

$$z = \int dP\left(\sqrt{P} \cdot f' : P - \frac{1}{2}\int \frac{dP}{\sqrt{P}} f' : P\right) \Pi' : P + \frac{1}{2} \Pi : P \int \frac{dP}{\sqrt{P}} f' : P - y\sqrt{P} \cdot \Pi : P;$$

at

$$\frac{1}{2} \Pi : P \int \frac{dP}{\sqrt{P}} f' : P = \int \left(\frac{1}{2} dP\, \Pi' : P \int \frac{dP}{\sqrt{P}} f' : P + \frac{1}{2} \frac{dP}{\sqrt{P}} \Pi : P f' : P\right),$$

unde fit

$$z = \int \Pi' : P\, dP \sqrt{P} \cdot f' : P + \frac{1}{2} \int \Pi : P \frac{dP}{\sqrt{P}} f' : P - y\sqrt{P} \cdot \Pi : P.$$

Porro est

$$\int dP\,\Pi':P\sqrt{P}\cdot f':P=\Pi:P\sqrt{P}\cdot f':P-\int \Pi:P\Big(\frac{dP}{2\sqrt{P}}f':P+dP\sqrt{P}\cdot f'':P\Big)$$

ideoque

$$z=\Pi:P\sqrt{P}\cdot f':P-\int dP\,\Pi:P\sqrt{P}\cdot f'':P-y\sqrt{P}\cdot\Pi:P;$$

statuatur porro

$$\int dP\,\Pi:P\sqrt{P}\cdot f'':P=\Theta:P;$$

erit $\Pi:P=\dfrac{\Theta':P}{\sqrt{P}\cdot f'':P}$ et

$$z=\frac{\Theta':P}{f'':P}(f':P-y)-\Theta:P+\Phi:\Big(\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\Big),$$

quae forma sine dubio multo est simplicior quam primo inventa.

PROBLEMA 50

312. *Proposita aequatione*

$$\Big(\frac{ddz}{dy^2}\Big)-PP\Big(\frac{ddz}{dx^2}\Big)+Q\Big(\frac{dz}{dy}\Big)+R\Big(\frac{dz}{dx}\Big)=0$$

invenire casus quantitatum P, Q, R, quibus integratio ope reductionis ante adhibitae succedit.

SOLUTIO

Introductis binis novis variabilibus t et u habebimus [§ 232]

$$\begin{aligned}0=\;&\Big(\frac{ddt}{dy^2}\Big)\Big(\frac{dz}{dt}\Big)+\Big(\frac{ddu}{dy^2}\Big)\Big(\frac{dz}{du}\Big)+\Big(\frac{dt}{dy}\Big)^2\Big(\frac{ddz}{dt^2}\Big)+2\Big(\frac{dt}{dy}\Big)\Big(\frac{du}{dy}\Big)\Big(\frac{ddz}{dt\,du}\Big)+\Big(\frac{du}{dy}\Big)^2\Big(\frac{ddz}{du^2}\Big)\\
&-P^2\Big(\frac{ddt}{dx^2}\Big)\quad -P^2\Big(\frac{ddu}{dx^2}\Big)\quad -P^2\Big(\frac{dt}{dx}\Big)^2\quad -2P^2\Big(\frac{dt}{dx}\Big)\Big(\frac{du}{dx}\Big)\quad -P^2\Big(\frac{du}{dx}\Big)^2\\
&+Q\Big(\frac{dt}{dy}\Big)\quad +Q\Big(\frac{du}{dy}\Big)\\
&+R\Big(\frac{dt}{dx}\Big)\quad +R\Big(\frac{du}{dx}\Big)\end{aligned}$$

Statuamus ergo ut ante

$$\left(\frac{dt}{dy}\right)=P\left(\frac{dt}{dx}\right) \quad \text{et} \quad \left(\frac{du}{dy}\right)=-P\left(\frac{du}{dx}\right),$$

unde fit

$$\left(\frac{ddt}{dxdy}\right)=P\left(\frac{ddt}{dx^2}\right)+\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)$$

et

$$\left(\frac{ddt}{dy^2}\right)=PP\left(\frac{ddt}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)+\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right)$$

atque

$$\left(\frac{ddu}{dy^2}\right)=PP\left(\frac{ddu}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{du}{dx}\right)-\left(\frac{dP}{dy}\right)\left(\frac{du}{dx}\right),$$

et aequatio resolvenda erit

$$0=\left(P\left(\frac{dP}{dx}\right)+\left(\frac{dP}{dy}\right)+PQ+R\right)\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)-4PP\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right)$$
$$+\left(P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)-PQ+R\right)\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right).$$

Iam evidens est integrationem institui posse, si alterutra formula $\left(\frac{dz}{dt}\right)$ vel $\left(\frac{dz}{du}\right)$ ex calculo abeat. Ponamus ergo esse

$$P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)-PQ+R=0 \quad \text{seu} \quad R=PQ+\left(\frac{dP}{dy}\right)-P\left(\frac{dP}{dx}\right)$$

et aequatio resultans per $\left(\frac{dt}{dx}\right)$ divisa fit

$$0=2\left(PQ+\left(\frac{dP}{dy}\right)\right)\left(\frac{dz}{dt}\right)-4PP\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right).$$

Fiat $\left(\frac{dz}{dt}\right)=v$; erit

$$\left(PQ+\left(\frac{dP}{dy}\right)\right)v-2PP\left(\frac{du}{dx}\right)\left(\frac{dv}{du}\right)=0;$$

sumatur t constans, ut fiat

$$\frac{dv}{v}=\frac{\left(PQ+\left(\frac{dP}{dy}\right)\right)du}{2PP\left(\frac{du}{dx}\right)},$$

ubi necesse est, ut quantitates P, Q, $\left(\frac{dP}{dy}\right)$ et $\left(\frac{du}{dx}\right)$ per novas variabiles t et u

exprimantur. Has ergo primum definiri convenit. Cum sit

$$\left(\frac{dt}{dy}\right) = P\left(\frac{dt}{dx}\right) \quad \text{et} \quad \left(\frac{du}{dy}\right) = -P\left(\frac{du}{dx}\right),$$

erit

$$dt = \left(\frac{dt}{dx}\right)(dx + Pdy) \quad \text{et} \quad du = \left(\frac{du}{dx}\right)(dx - Pdy);$$

sunt ergo $\left(\frac{dt}{dx}\right)$ et $\left(\frac{du}{dx}\right)$ factores integrabiles reddentes formulas $dx + Pdy$ et $dx - Pdy$; non enim opus est, ut hinc valores t et u generalissime definiantur. Sint p et q tales multiplicatores per x et y dati eritque

$$t = \int p(dx + Pdy) \quad \text{et} \quad u = \int q(dx - Pdy),$$

unde superior integratio fit

$$\frac{dv}{v} = \frac{\left(PQ + \left(\frac{dP}{dy}\right)\right)du}{2PPq},$$

in qua integratione quantitas $t = \int p(dx + Pdy)$ constans est spectanda. Seu ob $du = q(dx - Pdy)$ erit

$$\frac{dv}{v} = \frac{\left(PQ + \left(\frac{dP}{dy}\right)\right)(dx - Pdy)}{2PP}.$$

Verum ob $dt = 0$ est $dx = -Pdy$, ita ut prodeat

$$\frac{dv}{v} = -\frac{dy}{P}\left(PQ + \left(\frac{dP}{dy}\right)\right),$$

ubi ob t constans et datum per x et y valor ipsius x per y et t expressus substitui potest, ut sola y variabilis insit, et invento integrali

$$-\int \frac{dy}{P}\left(PQ + \left(\frac{dP}{dy}\right)\right) = lV$$

erit $v = Vf : t = \left(\frac{dz}{dt}\right)$. Nunc ponatur u constans; erit

$$z = \int Vdtf : t + F : u.$$

Conditio autem, sub qua haec integratio locum habet, postulat, ut sit

$$R = PQ + \left(\frac{dP}{dy}\right) - P\left(\frac{dP}{dx}\right).$$

COROLLARIUM 1

313. Eodem modo aequatio proposita resolutionem admittet, si fuerit

$$R = -PQ - \left(\frac{dP}{dy}\right) - P\left(\frac{dP}{dx}\right),$$

manetque ut ante

$$t = \int p(dx + Pdy) \quad \text{et} \quad u = \int q(dx - Pdy).$$

Tum vero fit

$$0 = -\left(PQ + \left(\frac{dP}{dy}\right)\right)\left(\frac{dz}{du}\right) - 2PP\left(\frac{dt}{dx}\right)\left(\frac{ddz}{dtdu}\right),$$

quae posito $\left(\frac{dz}{du}\right) = v$ sumtoque u constante dat

$$\frac{dv}{v} = \frac{-\left(PQ + \left(\frac{dP}{dy}\right)\right)dt}{2PP\left(\frac{dt}{dx}\right)} = \frac{-\left(PQ + \left(\frac{dP}{dy}\right)\right)(dx + Pdy)}{2PP}.$$

COROLLARIUM 2

314. Si porro habita ratione, quod $u = \int q(dx - Pdy)$ sit constans et $dx = Pdy$, ponatur

$$\int -\frac{dy\left(PQ + \left(\frac{dP}{dy}\right)\right)}{P} = lV,$$

erit

$$v = Vf : u = \left(\frac{dz}{du}\right),$$

unde tandem sumendo iam $t = \int p(dx + Pdy)$ [constans] colligitur

$$z = \int Vduf : u + F : t.$$

EXEMPLUM 1

315. *Si sumatur* $P = a$ *et* $R = aQ$, *quaecunque fuerit* Q *functio ipsarum* x *et* y, *integrare aequationem*

$$\left(\frac{ddz}{dy^2}\right) - aa\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dy}\right) + aQ\left(\frac{dz}{dx}\right) = 0.$$

Cum hic sit $P=a$, erit $p=1$, $q=1$ et $t=x+ay$ atque $u=x-ay$, unde posito $\left(\frac{dz}{dt}\right)=v$ fit

$$\frac{dv}{v}=\frac{aQdu}{2aa}=\frac{Qdu}{2a}.$$

Quoniam igitur est

$$x=\frac{t+u}{2} \quad \text{et} \quad y=\frac{t-u}{2a},$$

his valoribus substitutis fit Q functio ipsarum t et u ac spectata t ut constante erit

$$lv=\frac{1}{2a}\int Qdu+lf:t \quad \text{seu} \quad \left(\frac{dz}{dt}\right)=e^{\frac{1}{2a}\int Qdu}f:t$$

et sumta iam u constante

$$z=\int e^{\frac{1}{2a}\int Qdu}dtf:t+F:u.$$

COROLLARIUM 1

316. Si Q sit constans $=2ab$, aequationis huius

$$\left(\frac{ddz}{dy^2}\right)-aa\left(\frac{ddz}{dx^2}\right)+2ab\left(\frac{dz}{dy}\right)+2aab\left(\frac{dz}{dx}\right)=0$$

integrale erit

$$z=e^{bu}f:t+F:u=e^{b(x-ay)}f:(x+ay)+F:(x-ay)$$

sive

$$z=e^{b(x-ay)}(f:(x+ay)+F:(x-ay)).$$

COROLLARIUM 2

317. Si $Q=\frac{a}{x}$, huius aequationis

$$\left(\frac{ddz}{dy^2}\right)-aa\left(\frac{ddz}{dx^2}\right)+\frac{a}{x}\left(\frac{dz}{dy}\right)+\frac{aa}{x}\left(\frac{dz}{dx}\right)=0$$

integrale ob

$$\int Qdu=\int\frac{adu}{x}=\int\frac{2adu}{t+u}=2al(t+u)$$

erit

$$z=\int(t+u)dtf:t+F:u=\int tdtf:t+u\int dtf:t+F:u.$$

Vel sit $f:t = \Pi'':t$; erit $\int dt f:t = \Pi':t$ et

$$\int t dt f:t = \int t d.\Pi':t = t\Pi':t - \int dt \Pi':t = t\Pi':t - \Pi:t,$$

ergo

$$z = (t+u)\Pi':t - \Pi:t + F:u$$

seu

$$z = 2x\Pi':(x+ay) - \Pi:(x+ay) + F:(x-ay).$$

EXEMPLUM 2

318. *Sit* $P = \frac{x}{y}$ *et* $R = \frac{-x}{y}Q + \frac{x}{yy} - \frac{x}{yy} = \frac{-x}{y}Q$ *sumaturque* $Q = \frac{1}{x}$, *ut sit* $R = \frac{-1}{y}$ *et haec aequatio integrari debeat*

$$\left(\frac{ddz}{dy^2}\right) - \frac{xx}{yy}\left(\frac{ddz}{dx^2}\right) + \frac{1}{x}\left(\frac{dz}{dy}\right) - \frac{1}{y}\left(\frac{dz}{dx}\right) = 0.$$

Cum ergo sit

$$t = \int p\left(dx + \frac{xdy}{y}\right) \quad \text{et} \quad u = \int q\left(dx - \frac{xdy}{y}\right),$$

sumatur $p = y$ et $q = \frac{1}{y}$, ut fiat $t = xy$ et $u = \frac{x}{y}$. Posito nunc $\left(\frac{dz}{du}\right) = v$ sumtoque u constante ex Corollario 1 fit

$$\frac{dv}{v} = \frac{-\left(\frac{1}{y} - \frac{x}{yy}\right)dt}{\frac{2xx}{yy}y} = \frac{-(y-x)dt}{2xxy}.$$

Est vero $tu = xx$ hincque $x = \sqrt{tu}$ et $y = \sqrt{\frac{t}{u}}$ atque $2xxy = 2t\sqrt{tu}$, unde fit

$$\frac{dv}{v} = \frac{\left(\sqrt{tu} - \sqrt{\frac{t}{u}}\right)dt}{2t\sqrt{tu}} = \frac{dt}{2t} - \frac{dt}{2tu}$$

et ob u constans $lv = \frac{1}{2}lt - \frac{1}{2u}lt$, ergo

$$\left(\frac{dz}{du}\right) = t^{\frac{1}{2} - \frac{1}{2u}} f:u.$$

Quare sumto iam t constante erit

$$z = t^{\frac{1}{2}} \int t^{-\frac{1}{2u}} du f:u + F:t.$$

Vel ponatur $-\frac{1}{2u}=s$, ut sit $s=-\frac{y}{2x}$, eritque

$$z=t^{\frac{1}{2}}\int t^{\imath}dsf:s+F:t.$$

In hac integratione $\int t^{\imath}dsf:s$ sola s est variabilis ac demum integrali sumto restitui debet $t=xy$ et $s=\frac{-y}{2x}$.

Ceterum patet functionem quamcunque ipsius xy particulariter satisfacere.

PROBLEMA 51

319. *Proposita aequatione generali*

$$\left(\frac{ddz}{dy^2}\right)-2P\left(\frac{ddz}{dx\,dy}\right)+(PP-QQ)\left(\frac{ddz}{dx^2}\right)+R\left(\frac{dz}{dy}\right)+S\left(\frac{dz}{dx}\right)+Tz+V=0$$

invenire conditiones quantitatum P, Q, R, S, T, ut integratio ope reductionis adhibitae succedat.

SOLUTIO

Facta eadem substitutione [§ 232] introducendis binis novis variabilibus t et u aequatio nostra sequentem induet formam

$$0=V+Tz$$

$$\begin{array}{lllll}
+\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right) & +\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right) & +\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right) & +2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dt\,du}\right) & +\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right)\\
-2P\left(\frac{ddt}{dx\,dy}\right) & -2P\left(\frac{ddu}{dx\,dy}\right) & -2P\left(\frac{dt}{dx}\right)\left(\frac{dt}{dy}\right) & \begin{array}{l}-2P\left(\frac{dt}{dx}\right)\left(\frac{du}{dy}\right)\\ -2P\left(\frac{du}{dx}\right)\left(\frac{dt}{dy}\right)\end{array} & -2P\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)\\
+(P^2-Q^2)\left(\frac{ddt}{dx^2}\right) & +(P^2-Q^2)\left(\frac{ddu}{dx^2}\right) & +(P^2-Q^2)\left(\frac{dt}{dx}\right)^2 & +2(P^2-Q^2)\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right) & +(P^2-Q^2)\left(\frac{du}{dx}\right)^2\\
+R\left(\frac{dt}{dy}\right) & +R\left(\frac{du}{dy}\right) & & & \\
+S\left(\frac{dt}{dx}\right) & +S\left(\frac{du}{dx}\right) & & &
\end{array}$$

Determinentur iam hae duae novae variabiles t et u ita per x et y, ut formulae $\left(\frac{ddz}{dt^2}\right)$ et $\left(\frac{ddz}{du^2}\right)$ evanescant, debebitque esse

$$\left(\frac{dt}{dy}\right)=(P+Q)\left(\frac{dt}{dx}\right)\quad\text{et}\quad\left(\frac{du}{dy}\right)=(P-Q)\left(\frac{du}{dx}\right),$$

unde patet has variabiles sequenti modo determinari

$$t = \int p\big(dx + (P+Q)dy\big) \quad \text{et} \quad u = \int q\big(dx + (P-Q)dy\big)$$

sumendo p et q ita, ut hae formulae integrationem admittant. Cum nunc sit

$$\left(\frac{ddt}{dxdy}\right) = (P+Q)\left(\frac{ddt}{dx^2}\right) + \left(\left(\frac{dP}{dx}\right) + \left(\frac{dQ}{dx}\right)\right)\left(\frac{dt}{dx}\right),$$

$$\left(\frac{ddt}{dy^2}\right) = (P+Q)^2\left(\frac{ddt}{dx^2}\right) + (P+Q)\left(\left(\frac{dP}{dx}\right) + \left(\frac{dQ}{dx}\right)\right)\left(\frac{dt}{dx}\right) + \left(\left(\frac{dP}{dy}\right) + \left(\frac{dQ}{dy}\right)\right)\left(\frac{dt}{dx}\right),$$

$$\left(\frac{ddu}{dxdy}\right) = (P-Q)\left(\frac{ddu}{dx^2}\right) + \left(\left(\frac{dP}{dx}\right) - \left(\frac{dQ}{dx}\right)\right)\left(\frac{du}{dx}\right),$$

$$\left(\frac{ddu}{dy^2}\right) = (P-Q)^2\left(\frac{ddu}{dx^2}\right) + (P-Q)\left(\left(\frac{dP}{dx}\right) - \left(\frac{dQ}{dx}\right)\right)\left(\frac{du}{dx}\right) + \left(\left(\frac{dP}{dy}\right) - \left(\frac{dQ}{dy}\right)\right)\left(\frac{du}{dx}\right),$$

hinc reperitur formulae $2\left(\frac{ddz}{dtdu}\right)$ coefficiens

$$-2QQ\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right),$$

termini $\left(\frac{dz}{dt}\right)$ coefficiens

$$\left(-(P-Q)\left(\frac{dP+dQ}{dx}\right) + \left(\frac{dP+dQ}{dy}\right) + R(P+Q) + S\right)\left(\frac{dt}{dx}\right),$$

termini vero $\left(\frac{dz}{du}\right)$ coefficiens

$$\left(-(P+Q)\left(\frac{dP-dQ}{dx}\right) + \left(\frac{dP-dQ}{dy}\right) + R(P-Q) + S\right)\left(\frac{du}{dx}\right).$$

Est vero

$$\left(\frac{dt}{dx}\right) = p \quad \text{et} \quad \left(\frac{du}{dx}\right) = q,$$

unde, si brevitatis gratia vocetur

$$S + R(P+Q) + \left(\frac{dP+dQ}{dy}\right) - (P-Q)\left(\frac{dP+dQ}{dx}\right) = M$$

et

$$S + R(P-Q) + \left(\frac{dP-dQ}{dy}\right) - (P+Q)\left(\frac{dP-dQ}{dx}\right) = N,$$

aequatio nostra resolvenda erit

$$0 = V + Tz + Mp\left(\frac{dz}{dt}\right) + Nq\left(\frac{dz}{du}\right) - 4QQpq\left(\frac{ddz}{dtdu}\right)$$

seu, ut cum formis supra § 294 et 295 exhibitis comparari queat,

$$\left(\frac{ddz}{dtdu}\right)-\frac{M}{4QQq}\left(\frac{dz}{dt}\right)-\frac{N}{4QQp}\left(\frac{dz}{du}\right)-\frac{T}{4QQpq}z-\frac{V}{4QQpq}=0,$$

quae, si porro brevitatis gratia ponatur

$$\frac{M}{4QQq}=K \quad \text{et} \quad \frac{N}{4QQp}=L,$$

duplici casu integrationem admittit: altero, si fuerit

$$-\frac{T}{4QQpq}=KL-\left(\frac{dL}{du}\right) \quad \text{seu} \quad T=4QQpq\left(\frac{dL}{du}\right)-\frac{MN}{4QQ},$$

altero vero, si fuerit

$$-\frac{T}{4QQpq}=KL-\left(\frac{dK}{dt}\right) \quad \text{seu} \quad T=4QQpq\left(\frac{dK}{dt}\right)-\frac{MN}{4QQ}.$$

Quoniam vero K et L per x et y dantur, formulae illae $\left(\frac{dK}{dt}\right)$ et $\left(\frac{dL}{du}\right)$ ita reduci possunt, ut sit

$$\left(\frac{dK}{dt}\right)=\frac{Q-P}{2Qp}\left(\frac{dK}{dx}\right)+\frac{1}{2Qp}\left(\frac{dK}{dy}\right) \quad \text{et} \quad \left(\frac{dL}{du}\right)=\frac{P+Q}{2Qq}\left(\frac{dL}{dx}\right)-\frac{1}{2Qq}\left(\frac{dL}{dy}\right).$$

Quemadmodum autem ipsa integralia his casibus inveniri debeant, id quidem supra [§ 294, 295] est declaratum, unde superfluum foret calculos illos taediosos hic repetere; quovis enim casu oblato solutio inde peti poterit.

SCHOLION 1

320. Quod ad hanc reductionem formularum attinet, ea sequenti modo instituitur. Cum sit in genere

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right),$$

ex formulis

$$dt=pdx+p(P+Q)dy \quad \text{et} \quad du=qdx+q(P-Q)dy$$

erit

$$qdt-pdu=2pqQdy \quad \text{seu} \quad dy=\frac{qdt-pdu}{2Qpq}$$

et

$$q(P-Q)dt-p(P+Q)du=-2Qpqdx \quad \text{seu} \quad dx=\frac{p(P+Q)du-q(P-Q)dt}{2Qpq}.$$

Quibus valoribus substitutis obtinebitur

$$dz = \left(\frac{(P+Q)du}{2Qq} - \frac{(P-Q)dt}{2Qp}\right)\left(\frac{dz}{dx}\right) + \left(\frac{dt}{2Qp} - \frac{du}{2Qq}\right)\left(\frac{dz}{dy}\right),$$

ita ut dz per differentialia dt et du exprimatur. Posito ergo u constante et $du = 0$ erit

$$\left(\frac{dz}{dt}\right) = \frac{Q-P}{2Qp}\left(\frac{dz}{dx}\right) + \frac{1}{2Qp}\left(\frac{dz}{dy}\right),$$

at posito t constante et $dt = 0$ erit

$$\left(\frac{dz}{du}\right) = \frac{P+Q}{2Qq}\left(\frac{dz}{dx}\right) - \frac{1}{2Qq}\left(\frac{dz}{dy}\right).$$

SCHOLION 2

321. Methodus igitur hoc capite tradita in hoc consistit, ut huiusmodi aequationes ope introductionis binarum novarum variabilium t et u ad hanc formam reducantur

$$\left(\frac{ddz}{dtdu}\right) + P\left(\frac{dz}{dt}\right) + Q\left(\frac{dz}{du}\right) + Rz + S = 0,$$

de qua in praecedente capite [§ 275, 278, 287, 294, 295] vidimus, quibusnam casibus ea integrari queat. Iisdem igitur quoque casibus omnes aequationes, quae ad talem formam se reduci patiuntur, integrationem admittent.

Est vero eiusdem formae casus quidam maxime singularis, cuius integratio absolvi potest, unde denuo infinita multitudo aliarum aequationum, quae quidem eo reduci queant, oritur integrationem pariter admittentium. Quem propterea casum sequenti capite diligentius evolvamus.

CAPUT IV

ALIA METHODUS PECULIARIS HUIUSMODI AEQUATIONES INTEGRANDI

PROBLEMA 52

322. *Si aequatio proposita hanc habuerit formam*[1])

$$(x+y)^2\left(\frac{ddz}{dxdy}\right)+m(x+y)\left(\frac{dz}{dx}\right)+m(x+y)\left(\frac{dz}{dy}\right)+nz=0,$$

eius integrale completum investigare.

SOLUTIO

Cum hic binae variabiles x et y aequaliter insint, ponatur primo

$$z=A(x+y)^{\lambda}f:x+B(x+y)^{\lambda+1}f':x+C(x+y)^{\lambda+2}f'':x+D(x+y)^{\lambda+3}f''':x+\text{etc.},$$

ubi pro faciliori substitutione notetur posito $v=(x+y)^{\mu}F:x$ fore

$$\left(\frac{dv}{dx}\right)=\mu(x+y)^{\mu-1}F:x+(x+y)^{\mu}F':x,$$

$$\left(\frac{dv}{dy}\right)=\mu(x+y)^{\mu-1}F:x$$

et

$$\left(\frac{ddv}{dxdy}\right)=\mu(\mu-1)(x+y)^{\mu-2}F:x+\mu(x+y)^{\mu-1}F':x.$$

1) Formam generaliorem huiusmodi aequationum LAPLACE tractavit in Commentatione: *Recherches sur le Calcul intégral aux différences partielles,* Mém. de l'acad. d. sc. de Paris (1773) 1777, p. 341, vide praesertim p. 376; *Oeuvres de LAPLACE*, t. IX, Paris 1893, p. 5 et 41. F. E.

Facta ergo substitutione obtinebimus hanc aequationem

$$\begin{array}{llll}
0= & nA(x+y)^{\lambda}f{:}x+ & nB(x+y)^{\lambda+1}f'{:}x+ & nC(x+y)^{\lambda+2}f''{:}x+\text{etc.}\\
 & & +\quad mA & +\quad mB\\
 & +2m\lambda A & +2m(\lambda+1)B & +\ 2m(\lambda+2)C\\
 & & +\quad \lambda A & +\quad (\lambda+1)B\\
 & +\lambda(\lambda-1)A & +\ (\lambda+1)\lambda B & +(\lambda+2)(\lambda+1)C
\end{array}$$

ubi totum negotium ad coefficientium A, B, C, D etc. determinationem revocatur; facile autem erat praevidere forma superiori assumta potestates ipsius $(x+y)$ in singulis membris pares esse prodituras. Fieri igitur necesse est

$$\begin{gathered}
n+2m\lambda+\lambda\lambda-\lambda=0,\\
(n+2m\lambda+2m+\lambda\lambda+\lambda)B+(m+\lambda)A=0,\\
(n+2m\lambda+4m+\lambda\lambda+3\lambda+2)\,C+(m+\lambda+1)B=0,\\
(n+2m\lambda+6m+\lambda\lambda+5\lambda+6)D+(m+\lambda+2)\,C=0\\
\text{etc.,}
\end{gathered}$$

quae determinationes ope primae $n+2m\lambda+\lambda\lambda-\lambda=0$ ita commodius exprimuntur

$$B=-\frac{m+\lambda}{2m+2\lambda}A \qquad F=-\frac{m+\lambda+4}{5(2m+2\lambda+4)}E$$

$$C=-\frac{m+\lambda+1}{2(2m+2\lambda+1)}B \qquad G=-\frac{m+\lambda+5}{6(2m+2\lambda+5)}F$$

$$D=-\frac{m+\lambda+2}{3(2m+2\lambda+2)}C \qquad H=-\frac{m+\lambda+6}{7(2m+2\lambda+6)}G$$

$$E=-\frac{m+\lambda+3}{4(2m+2\lambda+3)}D \qquad \text{etc.,}$$

unde lex progressionis est manifesta.

At pro exponente λ duplicem eruimus valorem

$$\lambda=\frac{1}{2}-m\pm\sqrt{\left(\frac{1}{4}-m-n+mm\right)},$$

quorum utrumque aeque pro λ accipere licet. Hic autem praecipue notandi sunt casus, quibus series assumta abrumpitur, quod fit, quoties $m+\lambda+i=0$

denotante i numerum quemcunque integrum positivum cyphra non exclusa. Hoc ergo evenit, quoties fuerit

$$\frac{1}{2}+i\pm\sqrt{\left(\frac{1}{4}-m-n+mm\right)}=0,$$

id quod fieri nequit, nisi $\frac{1}{4}-m-n+mm$ fuerit quadratum. Inventa autem huiusmodi serie sive finita sive in infinitum excurrente alia similis pro functionibus ipsius y reperitur, unde valor ipsius z ita reperietur expressus

$$\begin{aligned}z=A(x+y)^{\lambda}(f:x+F:y)&+B(x+y)^{\lambda+1}(f':x+F':y)\\+C(x+y)^{\lambda+2}(f'':x+F'':y)&+D(x+y)^{\lambda+3}(f''':x+F''':y)\\+E(x+y)^{\lambda+4}(f^{\mathrm{IV}}:x+F^{\mathrm{IV}}:y)&+F(x+y)^{\lambda+5}(f^{\mathrm{V}}:x+F^{\mathrm{V}}:y)\\&\text{etc.};\end{aligned}$$

ubi cum binae functiones arbitrariae adsint, id certum est signum hanc formam esse integrale completum aequationis propositae.

COROLLARIUM 1

323. Si fuerit $\lambda=-m$, hoc est $n-mm+m=0$ seu $n=mm-m$, integrale ex unico membro constabit ob $B=0$[1]) eritque integrale

$$z=A(x+y)^{-m}(f:x+F:y).$$

COROLLARIUM 2

324. Integrale autem duo membra continebit, si

$$\lambda=-m-1 \quad \text{vel} \quad n=mm-m-2=(m+1)(m-2);$$

tum erit $B=-\frac{1}{2}A$ et integrale erit

$$z=(x+y)^{-m-1}(f:x+F:y)-\frac{1}{2}(x+y)^{-m}(f':x+F':y).$$

1) Revera coefficiens B manet arbitrarius. Cf. casum $k=0$ § 329. F. E.

COROLLARIUM 3

325. Integrale tribus terminis constabit, si

$$\lambda = -m-2 \quad \text{vel} \quad n=(m+2)(m-3);$$

tum erit

$$B=-\tfrac{1}{2}A \quad \text{et} \quad C=-\tfrac{1}{6}B=+\tfrac{1}{12}A,$$

integrale vero

$$z=(x+y)^{-m-2}(f:x+F:y)-\tfrac{1}{2}(x+y)^{-m-1}(f':x+F':y)$$
$$+\tfrac{1}{12}(x+y)^{-m}(f'':x+F'':y).$$

COROLLARIUM 4

326. Ex quatuor autem membris integrale constabit, si fuerit

$$\lambda=-m-3 \quad \text{seu} \quad n=(m+3)(m-4);$$

tum autem erit

$$B=-\tfrac{1}{2}A,\quad C=-\tfrac{1}{5}B=+\tfrac{1}{10}A,\quad D=-\tfrac{1}{12}C=-\tfrac{1}{120}A$$

et integrale

$$z=(x+y)^{-m-3}(f:x+F:y)-\tfrac{1}{2}(x+y)^{-m-2}(f':x+F':y)$$
$$+\tfrac{1}{10}(x+y)^{-m-1}(f'':x+F'':y)-\tfrac{1}{120}(x+y)^{-m}(f''':x+F''':y).$$

SCHOLION

327. Quodsi in genere ponamus $\lambda+m=-i$, erit $n=(m+i)(m-i-1)$, tum vero

$$B=-\frac{1}{2}A,\quad C=-\frac{i-1}{2(2i-1)}B,\quad D=-\frac{i-2}{3(2i-2)}C,\quad E=-\frac{i-3}{4(2i-3)}D,$$

unde fit omnes ad primum reducendo

$$B=-\frac{1}{2}A,\quad C=\frac{i-1}{2\cdot 2(2i-1)}A,\quad D=\frac{-(i-2)}{2\cdot 2\cdot 2\cdot 3(2i-1)}A,$$

$$E=\frac{+(i-2)(i-3)}{2\cdot 2\cdot 2\cdot 3\cdot 4(2i-1)(2i-3)}A,\quad F=\frac{-(i-3)(i-4)}{2^4\cdot 3\cdot 4\cdot 5(2i-1)(2i-3)}A \quad \text{etc.},$$

qui ita se habent:

	A	B	C	D	E	F	G
$i=1$	1	$-\frac{1}{2}$	0	0	0	0	0
$i=2$	1	$-\frac{1}{2}$	$\frac{1}{12}$	0	0	0	0
$i=3$	1	$-\frac{1}{2}$	$\frac{2}{20}$	$-\frac{1}{120}$	0	0	0
$i=4$	1	$-\frac{1}{2}$	$\frac{3}{28}$	$-\frac{2}{7\cdot 24}$	$\frac{2\cdot 1}{96\cdot 7\cdot 5}$	0	0
$i=5$	1	$-\frac{1}{2}$	$\frac{4}{36}$	$-\frac{3}{9\cdot 24}$	$\frac{3\cdot 2}{96\cdot 9\cdot 7}$	$-\frac{2\cdot 1}{960\cdot 9\cdot 7}$	0
$i=6$	1	$-\frac{1}{2}$	$\frac{5}{44}$	$-\frac{4}{11\cdot 24}$	$\frac{4\cdot 3}{96\cdot 11\cdot 9}$	$-\frac{3\cdot 2}{960\cdot 11\cdot 9}$	$\frac{3\cdot 2\cdot 1}{5760\cdot 11\cdot 9\cdot 7}$

Ita huius aequationis

$$\left(\frac{ddz}{dxdy}\right)+\frac{m}{x+y}\left(\frac{dz}{dx}\right)+\frac{m}{x+y}\left(\frac{dz}{dy}\right)+\frac{(m+i)(m-i-1)}{(x+y)^2}z=0$$

integrale completum erit

$$\begin{aligned}
z=&(x+y)^{-m-i}(f:x+F:y)\\
&-\frac{i}{2i}(x+y)^{-m-i+1}(f':x+F':y)\\
&+\frac{i(i-1)}{2i\cdot 2(2i-1)}(x+y)^{-m-i+2}(f'':x+F'':y)\\
&-\frac{i(i-1)(i-2)}{2i\cdot 2(2i-1)\cdot 3(2i-2)}(x+y)^{-m-i+3}(f''':x+F''':y)\\
&+\frac{i(i-1)(i-2)(i-3)}{2i\cdot 2(2i-1)\cdot 3(2i-2)\cdot 4(2i-3)}(x+y)^{-m-i+4}(f^{\text{IV}}:x+F^{\text{IV}}:y)\\
&-\frac{i(i-1)(i-2)(i-3)(i-4)}{2i\cdot 2(2i-1)\cdot 3(2i-2)\cdot 4(2i-3)\cdot 5(2i-4)}(x+y)^{-m-i+5}(f^{\text{V}}:x+F^{\text{V}}:y)\\
&\text{etc.},
\end{aligned}$$

quae forma, quoties i fuerit numerus integer positivus, finito constat terminorum numero; secus autem in infinitum excurrit.

Imprimis autem ista integratio hoc habet singulare, quod non solum ipsas functiones arbitrarias $f:x$ et $F:y$ complectatur, sed etiam earum formulas differentiales.

EXEMPLUM

328. *Si occurrat ista aequatio*

$$\left(\frac{ddz}{dxdy}\right)+\frac{m}{x+y}\left(\frac{dz}{dx}\right)+\frac{m}{x+y}\left(\frac{dz}{dy}\right)=0,$$

definire casus, quibus eius integrale per formam finitam exhiberi potest.

Cum hic sit $n=(m+i)(m-i-1)=0$, sumendo pro i numeros integros positivos duo ordines habebuntur casuum, quibus integratio succedit, alter, quo est $m=-i$, alter, quo $m=i+1$, ita ut in genere integratio finita locum habeat, quoties m fuerit numerus integer sive positivus sive negativus.

Primo ergo si sit $m=-i$, erit

$$\begin{aligned} z=1(f:x+F:y)&-\frac{i}{2i}(x+y)(f':x+F':y)\\ &+\frac{1}{2}\cdot\frac{i(i-1)}{2i(2i-1)}(x+y)^2(f'':x+F'':y)\\ &-\frac{1}{6}\cdot\frac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^3(f''':x+F''':y)\\ &+\frac{1}{24}\cdot\frac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^4(f^{\mathrm{IV}}:x+F^{\mathrm{IV}}:y)\\ &\text{etc.}\end{aligned}$$

Deinde si sit $m=i+1$, erit

$$\begin{aligned} (x+y)^{2i+1}z=1(f:x+F:y)&-\frac{i}{2i}(x+y)(f':x+F':y)\\ &+\frac{1}{2}\cdot\frac{i(i-1)}{2i(2i-1)}(x+y)^2(f'':x+F'':y)\\ &-\frac{1}{6}\cdot\frac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^3(f''':x+F''':y)\\ &+\frac{1}{24}\cdot\frac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^4(f^{\mathrm{IV}}:x+F^{\mathrm{IV}}:y)\\ &\text{etc.}\end{aligned}$$

Utrinque scilicet eadem habetur expressio, cui casu priori ipsa quantitas z, posteriori quantitas $(x+y)^{2i+1}z$ aequatur.

Ad singulos hos casus distinctius evolvendos ponamus

$$A = (f:x + F:y),$$
$$B = (f:x + F:y) - \frac{1}{2}(x+y)(f':x + F':y),$$
$$C = (f:x + F:y) - \frac{2}{4}(x+y)(f':x + F':y) + \frac{1}{4\cdot 3}(x+y)^2(f'':x + F'':y),$$
$$D = (f:x + F:y) - \frac{3}{6}(x+y)(f':x + F':y) + \frac{3}{6\cdot 5}(x+y)^2(f'':x + F'':y)$$
$$- \frac{1}{6\cdot 5\cdot 4}(x+y)^3(f''':x + F''':y)$$

etc.

vel posito brevitatis gratia

$$\mathfrak{A} = f:x + F:y,$$
$$\mathfrak{B} = (x+y)(f':x + F':y),$$
$$\mathfrak{C} = (x+y)^2(f'':x + F'':y),$$
$$\mathfrak{D} = (x+y)^3(f''':x + F''':y),$$
$$\mathfrak{E} = (x+y)^4(f^{\mathrm{IV}}:x + F^{\mathrm{IV}}:y)$$

etc.

sit

$$A = \mathfrak{A},$$
$$B = \mathfrak{A} - \frac{1}{2}\mathfrak{B},$$
$$C = \mathfrak{A} - \frac{2}{4}\mathfrak{B} + \frac{1}{4\cdot 3}\mathfrak{C},$$
$$D = \mathfrak{A} - \frac{3}{6}\mathfrak{B} + \frac{3}{6\cdot 5}\mathfrak{C} - \frac{1}{6\cdot 5\cdot 4}\mathfrak{D},$$
$$E = \mathfrak{A} - \frac{4}{8}\mathfrak{B} + \frac{6}{8\cdot 7}\mathfrak{C} - \frac{4}{8\cdot 7\cdot 6}\mathfrak{D} + \frac{1}{8\cdot 7\cdot 6\cdot 5}\mathfrak{E},$$
$$F = \mathfrak{A} - \frac{5}{10}\mathfrak{B} + \frac{10}{10\cdot 9}\mathfrak{C} - \frac{10}{10\cdot 9\cdot 8}\mathfrak{D} + \frac{5}{10\cdot 9\cdot 8\cdot 7}\mathfrak{E} - \frac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6}\mathfrak{F},$$
$$G = \mathfrak{A} - \frac{6}{12}\mathfrak{B} + \frac{15}{12\cdot 11}\mathfrak{C} - \frac{20}{12\cdot 11\cdot 10}\mathfrak{D} + \frac{15}{12\cdot 11\cdot 10\cdot 9}\mathfrak{E} - \frac{6}{12\cdot 11\cdot 10\cdot 9\cdot 8}\mathfrak{F}$$
$$+ \frac{1}{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}\mathfrak{G}$$

etc.

Quibus valoribus inventis erit pro duplici ordine:

si	erit	si	erit
$m = 0,$	$z = A$	$m = 1,$	$(x+y)\ z = A$
$m = -1,$	$z = B$	$m = 2,$	$(x+y)^3 z = B$
$m = -2,$	$z = C$	$m = 3,$	$(x+y)^5 z = C$
$m = -3,$	$z = D$	$m = 4,$	$(x+y)^7 z = D$
$m = -4,$	$z = E$	$m = 5,$	$(x+y)^9 z = E$
$m = -5,$	$z = F$	$m = 6,$	$(x+y)^{11} z = F$
$m = -6,$	$z = G$	$m = 7,$	$(x+y)^{13} z = G$
etc.		etc.	

SCHOLION

329. Si pro i sumatur numerus negativus, expressio in infinitum excurrit. Sit enim $i = -k$ et ex formula prima erit $m = k$ ideoque

$$z = \mathfrak{A} - \frac{k}{2k}\mathfrak{B} + \frac{1}{2}\cdot\frac{k(k+1)}{2k(2k+1)}\mathfrak{C} - \frac{1}{6}\cdot\frac{k(k+1)(k+2)}{2k(2k+1)(2k+2)}\mathfrak{D} + \text{etc. in infinitum.}$$

Pro eodem autem casu $m = k$ altera forma ob $i = k - 1$ dat

$$(x+y)^{2k-1} z$$
$$= \mathfrak{A} - \frac{k-1}{2k-2}\mathfrak{B} + \frac{1}{2}\cdot\frac{(k-1)(k-2)}{(2k-2)(2k-3)}\mathfrak{C} - \frac{1}{6}\cdot\frac{(k-1)(k-2)(k-3)}{(2k-2)(2k-3)(2k-4)}\mathfrak{D} + \text{etc.},$$

quae autem formae non absolute aequales sunt censendae, sed in altera functiones $f:x$ et $F:y$ alias formas habebunt, ut nihilominus ambae aeque satisfaciant. Casu quidem $k = \frac{1}{2}$ ambae conveniunt perfecte. Ponamus autem $k = 0$, ut prior det

$$z = \mathfrak{A} = f:x + F:y;$$

at posterior praebet

$$\frac{z}{x+y} = \mathfrak{A} - \frac{1}{2}\mathfrak{B} + \frac{1}{6}\mathfrak{C} - \frac{1}{24}\mathfrak{D} + \frac{1}{120}\mathfrak{E} - \text{etc.}$$

Quarum consensus ut appareat, sit in hac posteriori

$$f:x = ax^3 \quad \text{et} \quad F:y = by^2;$$

erit

$$\mathfrak{A} = ax^3 + by^2, \quad \mathfrak{B} = (x+y)(3axx + 2by), \quad \mathfrak{C} = (x+y)^2(6ax + 2b),$$
$$\mathfrak{D} = (x+y)^3 6a,$$

at reliquae partes evanescunt. Obtinebimus ergo ex posteriori

$$z = (x+y)(ax^3 + by^2) - \frac{1}{2}(x+y)^2(3axx + 2by)$$
$$+ \frac{1}{3}(x+y)^3(3ax + b) - \frac{1}{4}(x+y)^4 a,$$

quae evoluta praebet

$$\frac{1}{4}ax^4 - \frac{1}{4}ay^4 + \frac{1}{3}bx^3 + \frac{1}{3}by^3 = z,$$

quae forma utique in priori $z = f:x + F:y$ continetur. Consensus ergo binarum illarum formarum generalium eo magis est notatu dignus.

PROBLEMA 53

330. *Invenire casus, quibus haec aequatio generalis*

$$\left(\frac{ddz}{dy^2}\right) - QQ\left(\frac{ddz}{dx^2}\right) + R\left(\frac{dz}{dy}\right) + S\left(\frac{dz}{dx}\right) + Tz = 0$$

ad formam praecedentem reduci ideoque iisdem casibus integrari potest.

SOLUTIO

Introducendo binas novas variabiles t et u, ut sit, quemadmodum reductio § 319 adhibita, ubi $P = 0$ et $V = 0$, declarat,

$$t = \int p(dx + Qdy) \quad \text{et} \quad u = \int q(dx - Qdy),$$

si ponamus ad abbreviandum

$$M = S + QR + \left(\frac{dQ}{dy}\right) + Q\left(\frac{dQ}{dx}\right),$$

$$N = S - QR - \left(\frac{dQ}{dy}\right) + Q\left(\frac{dQ}{dx}\right),$$

prodibit haec aequatio

$$\left(\frac{ddz}{dtdu}\right)-\frac{M}{4QQq}\left(\frac{dz}{dt}\right)-\frac{N}{4QQp}\left(\frac{dz}{du}\right)-\frac{T}{4QQpq}z=0,$$

quam ergo ad hanc formam revocari oportet

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)+\frac{n}{(t+u)^2}z=0,$$

cuius casus integrabilitatis ante [§ 327] designavimus, scilicet quoties fuerit

$$n=(m+i)(m-i-1)$$

denotante i numerum integrum quemcunque positivum cyphra non exclusa.

Ad hoc ergo necesse est, ut fiat

$$M=\frac{-4mQQq}{t+u},\quad N=\frac{-4mQQp}{t+u}\quad \text{et}\quad T=\frac{-4nQQpq}{(t+u)^2}.$$

Quia autem hic integrabilitatis formularum t et u ratio haberi debet, sumamus

$$Q=\frac{\varphi':y}{\pi':x}$$

sitque

$$p=a\pi':x\quad \text{et}\quad q=b\pi':x$$

eritque

$$t=a\pi:x+a\varphi:y\quad \text{et}\quad u=b\pi:x-b\varphi:y.$$

Hinc fit

$$M+N=2S+2Q\left(\frac{dQ}{dx}\right)=\frac{-4m(a+b)QQ\pi':x}{t+u}$$

et

$$M-N=2QR+2\left(\frac{dQ}{dy}\right)=\frac{4m(a-b)QQ\pi':x}{t+u}$$

ideoque

$$R=\frac{2m(a-b)Q\pi':x}{t+u}-\frac{1}{Q}\left(\frac{dQ}{dy}\right),\quad S=\frac{-2m(a+b)QQ\pi':x}{t+u}-Q\left(\frac{dQ}{dx}\right)$$

et

$$T=\frac{-4nabQQ\pi':x\cdot\pi':x}{(t+u)^2}=\frac{-4nab\varphi':y\cdot\varphi':y}{(t+u)^2}$$

ob $Q = \frac{\varphi' : y}{\pi' : x}$, unde est

$$\left(\frac{dQ}{dy}\right) = \frac{\varphi'' : y}{\pi' : x} \quad \text{et} \quad \left(\frac{dQ}{dx}\right) = \frac{-\pi'' : x \cdot \varphi' : y}{\pi' : x \cdot \pi' : x}$$

et

$$t + u = (a+b)\pi : x + (a-b)\varphi : y.$$

Ideoque habebimus

$$R = \frac{2m(a-b)\varphi' : y}{t+u} - \frac{\varphi'' : y}{\varphi' : y} \quad \text{et} \quad \frac{S}{QQ} = \frac{-2m(a+b)\pi' : x}{t+u} + \frac{\pi'' : x}{\pi' : x}.$$

Quo aequatio fiat simplicior, duo casus praecipue sunt considerandi, alter, ubi $b = a$, alter $b = -a$. Priori est $t + u = 2a\pi : x$ et aequatio nostra erit

$$\left(\frac{ddz}{dy^2}\right) - \left(\frac{\varphi' : y}{\pi' : x}\right)^2 \left(\frac{ddz}{dx^2}\right)$$
$$- \frac{\varphi'' : y}{\varphi' : y}\left(\frac{dz}{dy}\right) + \left(\frac{\varphi' : y}{\pi' : x}\right)^2 \left(\frac{\pi'' : x}{\pi' : x} - \frac{2m\pi' : x}{\pi : x}\right)\left(\frac{dz}{dx}\right) - n\left(\frac{\varphi' : y}{\pi : x}\right)^2 z = 0;$$

altero vero casu $b = -a$ fit $t + u = 2a\varphi : y$ et

$$\left(\frac{ddz}{dy^2}\right) - \left(\frac{\varphi' : y}{\pi' : x}\right)^2 \left(\frac{ddz}{dx^2}\right)$$
$$+ \left(\frac{2m\varphi' : y}{\varphi : y} - \frac{\varphi'' : y}{\varphi' : y}\right)\left(\frac{dz}{dy}\right) + \left(\frac{\varphi' : y}{\pi' : x}\right)^2 \cdot \frac{\pi'' : x}{\pi' : x}\left(\frac{dz}{dx}\right) + n\left(\frac{\varphi' : y}{\varphi : y}\right)^2 z = 0,$$

quae ambae aequationes integrationem admittunt casibus

$$n = (m+i)(m-i-1).$$

COROLLARIUM 1

331. Aequationes postremo inventae a se invicem non differunt, nisi quod binae variabiles x et y invicem permutantur, unde sufficit alterutram solam considerasse. Prior autem [sumto $a = 1$] transformatur ponendo

$$t = \pi : x + \varphi : y \quad \text{et} \quad u = \pi : x - \varphi : y,$$

posterior vero ponendo

$$t = \pi : x + \varphi : y \quad \text{et} \quad u = \varphi : y - \pi : x.$$

COROLLARIUM 2

332. Hae aequationes etiam sequenti forma magis perspicua repraesentari possunt, prior quidem

$$\frac{1}{(\varphi':y)^2}\left(\frac{ddz}{dy^2}\right)-\frac{1}{(\pi':x)^2}\left(\frac{ddz}{dx^2}\right)$$
$$-\frac{\varphi'':y}{(\varphi':y)^3}\left(\frac{dz}{dy}\right)+\left(\frac{\pi'':x}{(\pi':x)^3}-\frac{2m}{\pi:x\cdot\pi':x}\right)\left(\frac{dz}{dx}\right)-\frac{n}{(\pi:x)^2}z=0$$

et posterior

$$\frac{1}{(\varphi':y)^2}\left(\frac{ddz}{dy^2}\right)-\frac{1}{(\pi':x)^2}\left(\frac{ddz}{dx^2}\right)$$
$$+\left(\frac{2m}{\varphi:y\cdot\varphi':y}-\frac{\varphi'':y}{(\varphi':y)^3}\right)\left(\frac{dz}{dy}\right)+\frac{\pi'':x}{(\pi':x)^3}\left(\frac{dz}{dx}\right)+\frac{n}{(\varphi:y)^2}z=0.$$

CASUS 1

333. Ponamus $\pi':x=a$ et $\varphi':y=b$; erit

$$\pi:x=ax \quad \text{et} \quad \varphi:y=by,$$

tum vero $\pi'':x=0$ et $\varphi'':y=0$, unde forma prior prodibit

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}\left(\frac{ddz}{dx^2}\right)-\frac{2m}{aax}\left(\frac{dz}{dx}\right)-\frac{n}{aaxx}z=0,$$

quae reducitur ad formam supra resolutam ponendo

$$t=ax+by \quad \text{et} \quad u=ax-by.$$

Posterior vero forma est

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{bby}\left(\frac{dz}{dy}\right)+\frac{n}{bbyy}z=0,$$

quae reducitur ad formam supra resolutam ponendo

$$t=ax+by \quad \text{et} \quad u=by-ax;$$

utraque autem est integrabilis casu

$$n=(m+i)(m-i-1).$$

Reductione enim ad variabiles t et u facta oritur haec aequatio

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)+\frac{n}{(t+u)^2}z=0.$$

COROLLARIUM 1

334. Si sumatur $n=0$, hae ambae aequationes

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right)-\left(\frac{ddz}{dx^2}\right)-\frac{2m}{x}\left(\frac{dz}{dx}\right)=0 \quad \text{et} \quad \left(\frac{ddz}{dy^2}\right)-\frac{bb}{aa}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{y}\left(\frac{dz}{dy}\right)=0$$

sunt integrabiles, quoties m fuerit numerus integer ideoque $2m$ numerus par.

COROLLARIUM 2

335. En ergo aequationes ob simplicitatem notatu dignas ex tribus tantum terminis constantes, quae infinitis casibus integrationem admittunt. Integrale autem quovis casu facile exhibetur ex § 328, si modo ibi loco x et y scribatur t et u.

CASUS 2

336. Sit $\pi':x=ax^\mu$ et $\varphi':y=b$; erit

$$\pi:x=\frac{1}{\mu+1}ax^{\mu+1} \quad \text{et} \quad \varphi:y=by,$$

tum vero $\pi'':x=\mu ax^{\mu-1}$ et $\varphi'':y=0$. Unde forma prior provenit

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)+\frac{\mu-2m\mu-2m}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+2}}z=0,$$

quae reducitur ad formam supra resolutam ponendo

$$t=\frac{1}{\mu+1}ax^{\mu+1}+by \quad \text{et} \quad u=\frac{1}{\mu+1}ax^{\mu+1}-by.$$

Posterior vero forma fit

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{bby}\left(\frac{dz}{dy}\right)+\frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)+\frac{n}{bbyy}z=0,$$

cuius reductio absolvitur ponendo

$$t = \frac{1}{\mu+1} ax^{\mu+1} + by \quad \text{et} \quad u = by - \frac{1}{\mu+1} ax^{\mu+1}.$$

Haeque ambae aequationes integrationem admittunt, quoties fuerit

$$n = (m+i)(m-i-1).$$

COROLLARIUM 1

337. Ex priori forma casus maxime notabilis existit, si capiatur $m = \frac{\mu}{2\mu+2}$ et $n = 0$; tum enim erit

$$\frac{aa}{bb} x^{2\mu} \left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right),$$

quae est integrabilis, quoties $\frac{\mu}{2\mu+2}$ fuerit numerus integer m sive positivus sive negativus.

COROLLARIUM 2

338. Vel cum sit $\mu = \frac{-2m}{2m-1}$, haec aequatio

$$\frac{aa}{bb} x^{\frac{-4m}{2m-1}} \left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) \quad \text{seu} \quad \left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{4m}{2m-1}} \left(\frac{ddz}{dx^2}\right)$$

erit integrabilis, quoties m fuerit numerus integer sive positivus sive negativus; reductio autem fit ponendo

$$t = -(2m-1) a x^{\frac{-1}{2m-1}} + by \quad \text{et} \quad u = -(2m-1) a x^{\frac{-1}{2m-1}} - by$$

CASUS 3

339. Sit $\pi' : x = ax^\mu$ et $\varphi' : y = by^\nu$; erit

$$\pi : x = \frac{1}{\mu+1} ax^{\mu+1} \quad \text{et} \quad \varphi : y = \frac{1}{\nu+1} by^{\nu+1},$$

tum vero $\pi'' : x = \mu a x^{\mu-1}$ et $\varphi'' : y = \nu b y^{\nu-1}$. Hinc prior forma resultat

$$\frac{1}{bby^{2\nu}} \left(\frac{ddz}{dy^2}\right) - \frac{1}{aax^{2\mu}} \left(\frac{ddz}{dx^2}\right) - \frac{\nu}{bby^{2\nu+1}} \left(\frac{dz}{dy}\right) + \frac{\mu - 2m\mu - 2m}{aax^{2\mu+1}} \left(\frac{dz}{dx}\right) - \frac{n(\mu+1)^2}{aax^{2\mu+2}} z = 0,$$

quae reducitur ponendo

$$t=\frac{1}{\mu+1}ax^{\mu+1}+\frac{1}{\nu+1}by^{\nu+1}\quad\text{et}\quad u=\frac{1}{\mu+1}ax^{\mu+1}-\frac{1}{\nu+1}by^{\nu+1}.$$

Posterior vero forma evadit

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)+\frac{2m\nu+2m-\nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)+\frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)+\frac{n(\nu+1)^2}{bby^{2\nu+2}}z=0,$$

cuius reductio fit hac substitutione

$$t=\frac{1}{\mu+1}ax^{\mu+1}+\frac{1}{\nu+1}by^{\nu+1}\quad\text{et}\quad u=\frac{-1}{\mu+1}ax^{\mu+1}+\frac{1}{\nu+1}by^{\nu+1}.$$

Vel cum hic tantum ratio inter a et b in computum ingrediatur, pro priori poni poterit

$$t=\frac{1}{2}x^{\mu+1}+\frac{(\mu+1)b}{2(\nu+1)a}y^{\nu+1}\quad\text{et}\quad u=\frac{1}{2}x^{\mu+1}-\frac{(\mu+1)b}{2(\nu+1)a}y^{\nu+1},$$

ut fiat $t+u=x^{\mu+1}$, quo expressio integralis fiat simplicior.

COROLLARIUM 1

340. Si ponatur in forma priori $\mu=\frac{-2m}{2m-1}$, minuetur ea uno termino fietque

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)-\frac{\nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)-\frac{n}{(2m-1)^2aa}x^{\frac{2}{2m-1}}z=0.$$

Statuatur $a=b$ et capiatur quoque $\nu=\frac{-2m}{2m-1}$, ut prodeat

$$y^{\frac{4m}{2m-1}}\left(\frac{ddz}{dy^2}\right)-x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{2m-1}y^{\frac{2m+1}{2m-1}}\left(\frac{dz}{dy}\right)-\frac{n}{(2m-1)^2}x^{\frac{2}{2m-1}}z=0.$$

COROLLARIUM 2

341. Sumatur porro in priori forma $\nu=\mu$ et fiat

$$\mu-2m\mu-2m=-\mu\quad\text{seu}\quad m=\frac{\mu}{\mu+1},$$

ut prodeat

$$\frac{1}{bby^{2\mu}}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)-\frac{\mu}{bby^{2\mu+1}}\left(\frac{dz}{dy}\right)-\frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+2}}z=0,$$

quae integrabilis existit, quoties fuerit

$$n = -\frac{(\mu + (\mu+1)i)((\mu+1)i+1)}{(\mu+1)^2} \quad \text{seu} \quad n = -\left(i + \frac{\mu}{\mu+1}\right)\left(i + \frac{1}{\mu+1}\right).$$

SCHOLION

342. Largissima ergo hinc nobis suppeditatur copia aequationum satis concinnarum, quas ope methodi hic traditae integrare licet. Atque hic imprimis duo casus conspiciuntur, quorum alter

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{4m}{2m-1}} \left(\frac{ddz}{dx^2}\right)$$

pro motu cordarum inaequali crassitie praeditarum determinando est inventus[1]), alter autem hac aequatione

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right) - \left(\frac{ddz}{dx^2}\right) - \frac{2m}{x}\left(\frac{dz}{dx}\right) = 0$$

contentus [§ 334] ideo est memorabilis, quod in analysi pro soni propagatione instituta ad talem formam pervenitur.[2]) Hae igitur binae aequationes prae ceteris merentur, ut pro casibus integrabilitatis integralia exhibeamus.

PROBLEMA 54

343. *Proposita aequatione differentiali*

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right) - \left(\frac{ddz}{dx^2}\right) - \frac{2m}{x}\left(\frac{dz}{dx}\right) = 0$$

casibus, quibus m est numerus integer sive positivus sive negativus, eius integrale completum exhibere.

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1) Vide ex. gr. L. EULERI Commentationes 287 et 442 (indicis ENESTROEMIANI): *De motu vibratorio cordarum inaequaliter crassarum*, Novi comment. acad. sc. Petrop. 9 (1762/3), 1764, p. 246, et *De motu vibratorio cordarum crassitie utcunque variabili praeditarum*, Novi comment. acad. sc. Petrop. 17 (1772), 1773, p. 432; *LEONHARDI EULERI Opera omnia*, series II, vol. 8 et 9. F. E.

2) Vide L. EULERI Commentationes 306, 307, 319 (indicis ENESTROEMIANI): *Supplément aux recherches sur la propagation du son*, Mém. de l'acad. d. sc. de Berlin [15] (1759), 1766, p. 210, *Continuation des recherches sur la propagation du son*, ibid. p. 241, *Recherches sur l'intégration de l'équation* $\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dx^2}\right) + \frac{b}{x}\left(\frac{dz}{dx}\right) + \frac{c}{xx}z$, Miscellanea Taurin. 3_2 (1762/5), 1766, p. 60; *LEONHARDI EULERI Opera omnia*, series III, vol. 1, et series I, vol. 23. F. E.

SOLUTIO

Facta substitutione $t=\frac{1}{2}x+\frac{b}{2a}y$ et $u=\frac{1}{2}x-\frac{b}{2a}y$ aequatio nostra hanc induit formam

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)=0.$$

Cum igitur sit $t+u=x$, si ponamus

$$\begin{aligned}
\mathfrak{A} &= \quad f\ :\frac{ax+by}{2a}+F\ :\frac{ax-by}{2a},\\
\mathfrak{B} &= x\left(f'\ :\frac{ax+by}{2a}+F'\ :\frac{ax-by}{2a}\right),\\
\mathfrak{C} &= x^2\left(f''\,:\frac{ax+by}{2a}+F''\,:\frac{ax-by}{2a}\right),\\
\mathfrak{D} &= x^3\left(f'''\!:\frac{ax+by}{2a}+F'''\!:\frac{ax-by}{2a}\right),\\
\mathfrak{E} &= x^4\left(f^{\mathrm{IV}}\!:\frac{ax+by}{2a}+F^{\mathrm{IV}}\!:\frac{ax-by}{2a}\right),\\
\mathfrak{F} &= x^5\left(f^{\mathrm{V}}:\frac{ax+by}{2a}+F^{\mathrm{V}}:\frac{ax-by}{2a}\right)
\end{aligned}$$

etc.,

casus integrabiles [§ 328] ita se habebunt, primo negativi:

si $m=0,\qquad z=\mathfrak{A},$

si $m=-1,\quad z=\mathfrak{A}-\frac{1}{2}\mathfrak{B},$

si $m=-2,\quad z=\mathfrak{A}-\frac{2}{4}\mathfrak{B}+\frac{1}{4\cdot 3}\mathfrak{C},$

si $m=-3,\quad z=\mathfrak{A}-\frac{3}{6}\mathfrak{B}+\frac{3}{6\cdot 5}\mathfrak{C}-\frac{1}{6\cdot 5\cdot 4}\mathfrak{D},$

si $m=-4,\quad z=\mathfrak{A}-\frac{4}{8}\mathfrak{B}+\frac{6}{8\cdot 7}\mathfrak{C}-\frac{4}{8\cdot 7\cdot 6}\mathfrak{D}+\frac{1}{8\cdot 7\cdot 6\cdot 5}\mathfrak{E},$

si $m=-5,\quad z=\mathfrak{A}-\frac{5}{10}\mathfrak{B}+\frac{10}{10\cdot 9}\mathfrak{C}-\frac{10}{10\cdot 9\cdot 8}\mathfrak{D}+\frac{5}{10\cdot 9\cdot 8\cdot 7}\mathfrak{E}-\frac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6}\mathfrak{F}$

etc.,

tum vero pro valoribus positivis ipsius m:

si $m=1,\quad x\ z=\mathfrak{A},$

si $m=2,\quad x^3 z=\mathfrak{A}-\frac{1}{2}\mathfrak{B},$

si $m=3,\quad x^5 z=\mathfrak{A}-\frac{2}{4}\mathfrak{B}+\frac{1}{4\cdot 3}\mathfrak{C},$

si $m=4,\quad x^7 z=\mathfrak{A}-\frac{3}{6}\mathfrak{B}+\frac{3}{6\cdot 5}\mathfrak{C}-\frac{1}{6\cdot 5\cdot 4}\mathfrak{D},$

si $m=5,\quad x^9 z=\mathfrak{A}-\frac{4}{8}\mathfrak{B}+\frac{6}{8\cdot 7}\mathfrak{C}-\frac{4}{8\cdot 7\cdot 6}\mathfrak{D}+\frac{1}{8\cdot 7\cdot 6\cdot 5}\mathfrak{E},$

si $m=6,\quad x^{11} z=\mathfrak{A}-\frac{5}{10}\mathfrak{B}+\frac{10}{10\cdot 9}\mathfrak{C}-\frac{10}{10\cdot 9\cdot 8}\mathfrak{D}+\frac{5}{10\cdot 9\cdot 8\cdot 7}\mathfrak{E}-\frac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6}\mathfrak{F}$

etc.

Cui ergo expressioni casu $m=-i$ aequatur valor z, eidem aequatur casu $m=i+1$ valor ipsius $x^{2i+1}z$.

SCHOLION

344. Valores ipsarum t et u ita hic assumsi, ut fieret $t+u=x$, atque eosdem valores quoque in functionibus adhiberi oportet. Etsi enim $f:\frac{ax+by}{2a}$ etiam est functio ipsius $ax+by$, tamen functiones per differentiationem inde derivatae discrepant. Namque si ponamus

$$f:\frac{ax+by}{2a}=\varphi:(ax+by),$$

erit differentiando

$$\frac{adx+bdy}{2a}f':\frac{ax+by}{2a}=(adx+bdy)\varphi':(ax+by),$$

unde erit

$$f':\frac{ax+by}{2a}=2a\varphi':(ax+by)$$

neque ergo hae functiones differentiales sunt aequales, etiamsi principales assumtae sint aequales. Simili modo erit

$$f'':\frac{ax+by}{2a}=4aa\varphi'':(ax+by)$$

et

$$f''':\frac{ax+by}{2a}=8a^3\varphi''':(ax+by)$$

et ita porro.

PROBLEMA 55

345. *Proposita aequatione differentiali*

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)$$

casibus, quibus m est numerus integer sive positivus sive negativus, integrale completum exhibere.

SOLUTIO

Introductis novis variabilibus t et u, ita ut sit [§ 338]

$$t=\frac{1}{2}x^{\frac{-1}{2m-1}}-\frac{b}{2(2m-1)a}y \quad \text{et} \quad u=\frac{1}{2}x^{\frac{-1}{2m-1}}+\frac{b}{2(2m-1)a}y,$$

aequatio nostra hanc induit formam

$$\left(\frac{ddz}{dt\,du}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)=0,$$

ubi est

$$t+u=x^{\frac{-1}{2m-1}}.$$

Posito igitur

$$\mathfrak{A}=f:t+F:u, \qquad \mathfrak{B}=x^{\frac{-1}{2m-1}}(f':t+F':u),$$

$$\mathfrak{C}=x^{\frac{-2}{2m-1}}(f'':t+F'':u), \qquad \mathfrak{D}=x^{\frac{-3}{2m-1}}(f''':t+F''':u),$$

$$\mathfrak{E}=x^{\frac{-4}{2m-1}}(f^{\mathrm{IV}}:t+F^{\mathrm{IV}}:u), \qquad \mathfrak{F}=x^{\frac{-5}{2m-1}}(f^{\mathrm{V}}:t+F^{\mathrm{V}}:u)$$

etc.

percurramus primo casus, quibus m a cyphra per numeros negativos decrescit.

I. Si $m = 0$, erit aequationis

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}\left(\frac{ddz}{dx^2}\right)$$

integrale

$$z = f:\left(\frac{1}{2}x + \frac{b}{2a}y\right) + F:\left(\frac{1}{2}x - \frac{b}{2a}y\right).$$

II. Si $m = -1$, ob

$$t = \frac{1}{2}x^{\frac{1}{3}} + \frac{b}{6a}y \quad \text{et} \quad u = \frac{1}{2}x^{\frac{1}{3}} - \frac{b}{6a}y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^{\frac{4}{3}}\left(\frac{ddz}{dx^2}\right)$$

integrale

$$z = f:t + F:u - \frac{1}{2}x^{\frac{1}{3}}(f':t + F':u).$$

III. Si $m = -2$, ob

$$t = \frac{1}{2}x^{\frac{1}{5}} + \frac{b}{10a}y \quad \text{et} \quad u = \frac{1}{2}x^{\frac{1}{5}} - \frac{b}{10a}y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^{\frac{8}{5}}\left(\frac{ddz}{dx^2}\right)$$

integrale

$$z = f:t + F:u - \frac{2}{4}x^{\frac{1}{5}}(f':t + F':u) + \frac{1}{4\cdot 3}x^{\frac{2}{5}}(f'':t + F'':u).$$

IV. Si $m = -3$, ob

$$t = \frac{1}{2}x^{\frac{1}{7}} + \frac{b}{14a}y \quad \text{et} \quad u = \frac{1}{2}x^{\frac{1}{7}} - \frac{b}{14a}y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^{\frac{12}{7}}\left(\frac{ddz}{dx^2}\right)$$

integrale

$$z = f:t + F:u - \frac{3}{6}x^{\frac{1}{7}}(f':t + F':u) + \frac{3}{6\cdot 5}x^{\frac{2}{7}}(f'':t + F'':u)$$
$$- \frac{1}{6\cdot 5\cdot 4}x^{\frac{3}{7}}(f''':t + F''':u).$$

V. Si $m = -4$, ob

$$t = \frac{1}{2} x^{\frac{1}{9}} + \frac{b}{18a} y \quad \text{et} \quad u = \frac{1}{2} x^{\frac{1}{9}} - \frac{b}{18a} y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{16}{9}} \left(\frac{ddz}{dx^2}\right)$$

integrale

$$z = f:t + F:u - \frac{4}{8} x^{\frac{1}{9}}(f':t + F':u) + \frac{6}{8 \cdot 7} x^{\frac{2}{9}}(f'':t + F'':u)$$
$$- \frac{4}{8 \cdot 7 \cdot 6} x^{\frac{3}{9}}(f''':t + F''':u) + \frac{1}{8 \cdot 7 \cdot 6 \cdot 5} x^{\frac{4}{9}}(f^{\text{IV}}:t + F^{\text{IV}}:u)$$

et ita porro.

Pro altero vero casu, ubi m habet valores positivos, integralia sequenti modo exprimentur:

I. Si sit $m = 1$ seu

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^4 \left(\frac{ddz}{dx^2}\right),$$

ob

$$t = \frac{1}{2} x^{-1} - \frac{b}{2a} y \quad \text{et} \quad u = \frac{1}{2} x^{-1} + \frac{b}{2a} y$$

erit integrale

$$x^{-1} z = f:t + F:u \quad \text{seu} \quad z = x(f:t + F:u).$$

II. Si sit $m = 2$ seu

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{8}{3}} \left(\frac{ddz}{dx^2}\right),$$

ob

$$t = \frac{1}{2} x^{-\frac{1}{3}} - \frac{b}{6a} y \quad \text{et} \quad u = \frac{1}{2} x^{-\frac{1}{3}} + \frac{b}{6a} y$$

erit integrale

$$z = x(f:t + F:u) - \frac{1}{2} x^{\frac{2}{3}}(f':t + F':u).$$

III. Si sit $m = 3$ seu

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{12}{5}} \left(\frac{ddz}{dx^2}\right),$$

ob

$$t = \frac{1}{2} x^{-\frac{1}{5}} - \frac{b}{10a} y \quad \text{et} \quad u = \frac{1}{2} x^{-\frac{1}{5}} + \frac{b}{10a} y$$

erit integrale

$$z = x(f:t + F:u) - \frac{2}{4}x^{\frac{4}{5}}(f':t + F':u) + \frac{1}{4\cdot 3}x^{\frac{3}{5}}(f'':t + F'':u).$$

IV. Si sit $m = 4$ seu

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^{\frac{16}{7}}\left(\frac{ddz}{dx^2}\right),$$

ob

$$t = \frac{1}{2}x^{-\frac{1}{7}} - \frac{b}{14a}y \quad \text{et} \quad u = \frac{1}{2}x^{-\frac{1}{7}} + \frac{b}{14a}y$$

erit integrale

$$z = x(f:t + F:u) - \frac{3}{6}x^{\frac{6}{7}}(f':t + F':u) + \frac{3}{6\cdot 5}x^{\frac{5}{7}}(f'':t + F'':u)$$
$$- \frac{1}{6\cdot 5\cdot 4}x^{\frac{4}{7}}(f''':t + F''':u).$$

V. Si sit $m = 5$ seu

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^{\frac{20}{9}}\left(\frac{ddz}{dx^2}\right),$$

ob

$$t = \frac{1}{2}x^{-\frac{1}{9}} - \frac{b}{18a}y \quad \text{et} \quad u = \frac{1}{2}x^{-\frac{1}{9}} + \frac{b}{18a}y$$

erit integrale

$$z = x(f:t + F:u) - \frac{4}{8}x^{\frac{8}{9}}(f':t + F':u) + \frac{6}{8\cdot 7}x^{\frac{7}{9}}(f'':t + F'':u)$$
$$- \frac{4}{8\cdot 7\cdot 6}x^{\frac{6}{9}}(f''':t + F''':u) + \frac{1}{8\cdot 7\cdot 6\cdot 5}x^{\frac{5}{9}}(f^{\text{IV}}:t + F^{\text{IV}}:u)$$
etc.,

unde lex, qua has expressiones ulterius continuare licet, per se est manifesta.

SCHOLION 1

346. Casus isti integrabilitatis congruunt cum iis, qui in aequatione RICCATIANA dicta deprehenduntur; novimus scilicet aequationem hanc

$$dy + yydx = ax^{\frac{-4m}{2m-1}}dx$$

integrari posse, quoties m est numerus integer sive positivus sive negativus.[1]) Haec autem aequatio haud levi vinculo cum nostra forma est connexa, quod ita ostendi potest.

Proposita forma generali

$$\left(\frac{ddz}{dy^2}\right) = X\left(\frac{ddz}{dx^2}\right)$$

pro integralibus particularibus inveniendis statuatur $z = e^{\alpha y}v$, ut v sit functio ipsius x tantum; erit

$$\left(\frac{dz}{dx}\right) = e^{\alpha y}\cdot\frac{dv}{dx} \quad \text{et} \quad \left(\frac{ddz}{dx^2}\right) = e^{\alpha y}\cdot\frac{ddv}{dx^2};$$

tum vero $\left(\frac{ddz}{dy^2}\right) = \alpha\alpha e^{\alpha y}v$, unde prodit haec aequatio

$$\alpha\alpha v = \frac{X\,ddv}{dx^2};$$

in qua si porro statuatur $v = e^{\int p\,dx}$, oritur

$$\frac{\alpha\alpha dx}{X} = dp + ppdx,$$

ac si $X = Ax^{\frac{4m}{2m-1}}$ ut in nostro casu, haec aequatio fit

$$dp + ppdx = ax^{\frac{-4m}{2m-1}}dx.$$

Haud temere igitur evenire putandum est, quod utraque aequatio iisdem casibus integrationem admittat.

Interim tamen notatu dignum occurrit, quod casus $m = \infty$, qui in forma RICCATIANA fit facillimus, idem in nostra aequatione neutiquam integrationem admittat. Habetur quippe haec aequatio

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}xx\left(\frac{ddz}{dx^2}\right),$$

cuius reductio modo supra § 330 adhibito non succedit. Nam ob

$$Q = \frac{bx}{a}, \quad R = 0, \quad S = 0 \quad \text{et} \quad T = 0$$

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1) Vide L. EULERI *Institutionum calculi integralis* vol. I, § 436, vol. II, § 943, LEONHARDI EULERI *Opera omnia*, series I, vol. 11, p. 276, vol. 12, p. 160. F. E.

pro novis variabilibus ponitur

$$t = \int p\left(dx + \frac{bx\,dy}{a}\right) \quad \text{et} \quad u = \int q\left(dx - \frac{bx\,dy}{a}\right),$$

unde ob $M = \frac{bbx}{aa} = N$ oritur haec aequatio

$$\left(\frac{ddz}{dt\,du}\right) - \frac{1}{4qx}\left(\frac{dz}{dt}\right) - \frac{1}{4px}\left(\frac{dz}{du}\right) = 0,$$

quae sumendo

$$p = \frac{1}{x} \quad \text{et} \quad q = \frac{1}{x},$$

ut sit

$$t = lx + \frac{by}{a} \quad \text{et} \quad u = lx - \frac{by}{a},$$

transit in

$$\left(\frac{ddz}{dt\,du}\right) - \frac{1}{4}\left(\frac{dz}{dt}\right) - \frac{1}{4}\left(\frac{dz}{du}\right) = 0,$$

cuius integratio haud perspicitur.

SCHOLION 2

347. Aequationis autem $\left(\frac{ddz}{dy^2}\right) = xx\left(\frac{ddz}{dx^2}\right)$ integralia particularia infinita exhibere licet in hac forma $z = Ax^\lambda e^{\mu y}$ contenta. Cum enim hinc sit

$$\left(\frac{dz}{dy}\right) = \mu Ax^\lambda e^{\mu y} \quad \text{et} \quad \left(\frac{dz}{dx}\right) = \lambda Ax^{\lambda-1}e^{\mu y},$$

erit

$$\mu\mu Ax^\lambda e^{\mu y} = \lambda(\lambda-1)Ax^\lambda e^{\mu y}$$

ideoque $\mu = \sqrt{\lambda(\lambda-1)}$, unde ex quovis numero pro λ assumto bini valores pro μ oriuntur, ita ut habeatur

$$z = Ax^\lambda e^{y\sqrt{\lambda(\lambda-1)}} + Bx^\lambda e^{-y\sqrt{\lambda(\lambda-1)}},$$

et huiusmodi membrorum numerus variando λ in infinitum multiplicari potest.

Interim tamen singula haec membra adhuc generaliora reddi possunt. Posito enim $z = x^\lambda e^{\mu y} v$ videamus, an v necessario constans esse debeat; hinc autem fit

$$\left(\frac{dz}{dy}\right) = \mu x^\lambda e^{\mu y} v + x^\lambda e^{\mu y}\left(\frac{dv}{dy}\right) \quad \text{et} \quad \left(\frac{dz}{dx}\right) = \lambda x^{\lambda-1} e^{\mu y} v + x^\lambda e^{\mu y}\left(\frac{dv}{dx}\right)$$

ideoque nostra aequatio praebet per $x^\lambda e^{\mu y}$ divisa

$$\mu\mu v + 2\mu\left(\frac{dv}{dy}\right) + \left(\frac{ddv}{dy^2}\right) = \lambda(\lambda-1)v + 2\lambda x\left(\frac{dv}{dx}\right) + xx\left(\frac{ddv}{dx^2}\right).$$

Statuatur ut ante $\mu\mu = \lambda(\lambda-1)$ sitque $v = \alpha lx + \beta y$; erit

$$2\beta\mu = 2\alpha\lambda - \alpha \quad \text{seu} \quad \frac{\alpha}{\beta} = \frac{2\mu}{2\lambda-1} = \frac{2\sqrt{\lambda(\lambda-1)}}{2\lambda-1},$$

unde cuiusque membri ex numero λ nati forma erit

$$z = x^\lambda \left\{ \begin{array}{l} \quad e^{y\sqrt{\lambda(\lambda-1)}}\left(A + \dfrac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{A}}\, lx + \dfrac{2\lambda-1}{\mathfrak{A}}\, y\right) \\ + e^{-y\sqrt{\lambda(\lambda-1)}}\left(B - \dfrac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{B}}\, lx + \dfrac{2\lambda-1}{\mathfrak{B}}\, y\right) \end{array} \right\}.$$

Quomodocunque igitur non solum exponens λ, sed etiam quantitates A, $\mathfrak{A}$, B, $\mathfrak{B}$ varientur, infinita huiusmodi membra formari possunt, quae omnia iunctim sumta valorem completum functionis z praebere sunt censenda.

Quin etiam pro λ imaginaria assumi possunt; posito enim $\lambda = a + b\sqrt{-1}$ fit $\mu = p + q\sqrt{-1}$ existente

$$pp - qq = aa - a - bb \quad \text{et} \quad pp + qq = \sqrt{(aa+bb)(aa-2a+1+bb)},$$

tum vero est

$$x^\lambda = x^a(\cos. blx + \sqrt{-1}\cdot \sin. blx) \quad \text{et} \quad e^{\mu y} = e^{py}(\cos. qy + \sqrt{-1}\cdot \sin. qy),$$

unde colligitur forma realis

$$z = x^a e^{py} \left\{ \begin{array}{l} \quad A \cos.(blx+qy) + B(2plx + (2a-1)y)\cos.(blx+qy) \\ \qquad\qquad\qquad\qquad - B(2qlx + 2by)\sin.(blx+qy) \\ + \mathfrak{A}\sin.(blx+qy) + \mathfrak{B}(2plx + (2a-1)y)\sin.(blx+qy) \\ \qquad\qquad\qquad\qquad + \mathfrak{B}(2qlx + 2by)\cos.(blx+qy) \end{array} \right\}$$

ubi quantitates a et b pro lubitu assumere licet, unde simul p et q definiuntur.

Quodsi hic litteras b et q ut datas spectemus, binae reliquae a et p ex iis ita determinantur, ut sit

$$2a-1=q\sqrt{\left(\frac{1}{qq-bb}-4\right)} \quad \text{et} \quad p=\frac{b}{2}\sqrt{\left(\frac{1}{qq-bb}-4\right)};$$

hic ergo necesse est sit $qq>bb$ et $qq<bb+\frac{1}{4}$, seu qq inter hos arctos limites bb et $bb+\frac{1}{4}$ contineri debet. Statuatur

$$q=c \quad \text{et} \quad \sqrt{\left(\frac{1}{qq-bb}-4\right)}=2f,$$

ut sit

$$\frac{1}{qq-bb}=4(1+ff) \quad \text{seu} \quad cc-bb=\frac{1}{4(1+ff)}$$

atque

$$2a-1=2cf \quad \text{et} \quad p=bf,$$

ex quo forma integralium particularium erit

$$z=x^{cf+\frac{1}{2}}e^{bfy}\left\{\begin{array}{r} A\cos.(blx+cy)+2Bf(blx+cy)\cos.(blx+cy) \\ -2B(clx+by)\sin.(blx+cy) \\ +\mathfrak{A}\sin.(blx+cy)+2\mathfrak{B}f(blx+cy)\sin.(blx+cy) \\ +2\mathfrak{B}(clx+by)\cos.(blx+cy) \end{array}\right\}$$

quae posito brevitatis gratia angulo $blx+cy=\varphi$ transformatur in hanc

$$z=x^{cf+\frac{1}{2}}e^{bfy}(A\cos.(\varphi+\alpha)+Bf(blx+cy)\sin.(\varphi+\beta)+B(clx+by)\cos.(\varphi+\beta)),$$

ubi quantitates b, c, A, B, α, β ab arbitrio nostro pendent.

SCHOLION 3

348. Resolutio ergo aequationis

$$\left(\frac{ddz}{dy^2}\right)=xx\left(\frac{ddz}{dx^2}\right)$$

ita institui potest, ut fingatur

$$z=x^{\lambda}e^{\mu y}(mlx+ny),$$

unde fit

$$\left(\frac{dz}{dx}\right) = \lambda x^{\lambda-1} e^{\mu y}(mlx + ny) + mx^{\lambda-1} e^{\mu y}$$

et

$$\left(\frac{dz}{dy}\right) = \mu x^{\lambda} e^{\mu y}(mlx + ny) + nx^{\lambda} e^{\mu y}$$

hincque ulterius differentiando

$$\left(\frac{ddz}{dx^2}\right) = x^{\lambda-2} e^{\mu y}(m(2\lambda - 1) + \lambda(\lambda - 1)mlx + \lambda(\lambda - 1)ny)$$

et

$$\left(\frac{ddz}{dy^2}\right) = x^{\lambda} e^{\mu y}(2\mu n + \mu\mu mlx + \mu\mu ny).$$

Ex quo colligitur primo $\mu = \sqrt{\lambda(\lambda - 1)}$, deinde $2n\sqrt{\lambda(\lambda - 1)} = m(2\lambda - 1)$, ut sit $\frac{m}{n} = \frac{2\sqrt{\lambda(\lambda - 1)}}{2\lambda - 1}$, sicque eadem prodit integratio, quam modo ante dedimus.

CAPUT V

TRANSFORMATIO SINGULARIS EARUNDEM AEQUATIONUM

PROBLEMA 56

349. *Proposita hac aequatione*

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz,$$

in qua P, Q, R *sint functiones ipsius* x *tantum, eam ope substitutionis*

$$z = M\left(\frac{dv}{dx}\right) + Nv,$$

ubi quoque sint M *et* N *functiones ipsius* x *tantum, in aliam eiusdem formae transmutare, ut prodeat*

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

existentibus F, G, H *functionibus solius* x.

SOLUTIO

Quia quantitates M et N ab y sunt immunes, erit

$$\left(\frac{ddz}{dy^2}\right) = M\left(\frac{d^3v}{dx\,dy^2}\right) + N\left(\frac{ddv}{dy^2}\right),$$

quae forma per aequationem, quam tandem resultare assumimus, abit in hanc

$$\left(\frac{ddz}{dy^2}\right) = MF\left(\frac{d^3v}{dx^3}\right) + \begin{array}{l}\dfrac{MdF}{dx} \\ +MG \\ +NF\end{array}\left(\frac{ddv}{dx^2}\right) + \begin{array}{l}\dfrac{MdG}{dx} \\ +MH \\ +NG\end{array}\left(\frac{dv}{dx}\right) + \begin{array}{l}\dfrac{MdH}{dx} \\ +NH\end{array} v$$

Deinde vero pro altero aequationis propositae membro nostra substitutio praebet

$$\left(\frac{dz}{dx}\right) = M\left(\frac{ddv}{dx^2}\right) + \frac{dM}{dx}\left(\frac{dv}{dx}\right) + \begin{array}{l}\dfrac{dN}{dx} \\ +N\end{array} v$$

hincque porro

$$\left(\frac{ddz}{dx^2}\right) = M\left(\frac{d^3v}{dx^3}\right) + \left(\frac{2\,dM}{dx} + N\right)\left(\frac{ddv}{dx^2}\right) + \left(\frac{ddM}{dx^2} + \frac{2\,dN}{dx}\right)\left(\frac{dv}{dx}\right) + \frac{ddN}{dx^2}v.$$

Cum nunc sit per hypothesin

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz,$$

si hic valores modo inventi substituantur singulaque membra $\left(\frac{d^3v}{dx^3}\right)$, $\left(\frac{ddv}{dx^2}\right)$, $\left(\frac{dv}{dx}\right)$ et v seorsim ad nihilum redigantur, quatuor sequentes aequationes orientur, scilicet

ex	colligitur aequatio
$\left(\frac{d^3v}{dx^3}\right)$	$MF = MP,$
$\left(\frac{ddv}{dx^2}\right)$	$\frac{MdF}{dx} + MG + NF = \left(\frac{2dM}{dx} + N\right)P + MQ,$
$\left(\frac{dv}{dx}\right)$	$\frac{MdG}{dx} + MH + NG = \left(\frac{ddM}{dx^2} + \frac{2dN}{dx}\right)P + \left(\frac{dM}{dx} + N\right)Q + MR,$
v	$\frac{MdH}{dx} + NH = \frac{ddN}{dx^2}P + \frac{dN}{dx}Q + NR,$

ex quibus commodissime primo quaeruntur P, Q et R.

Verum prima dat statim $P = F$, unde secunda fit

$$\frac{MdF - 2FdM}{Mdx} + G = Q.$$

Ex binis ultimis autem eliminando R colligitur

$$\frac{M(NdG - MdH)}{dx} + NNG = \left(\frac{NddM - MddN}{dx^2} + \frac{2NdN}{dx}\right)F$$
$$+ \left(\frac{NdM - MdN}{dx} + NN\right)Q$$

et illum valorem pro Q substituendo

$$0 = \frac{MMdH}{dx} - \frac{MNdG}{dx} + \frac{NddM - MddN}{dx^2}F + \frac{2NFdN}{dx} + \frac{NdM - MdN}{dx}G$$
$$+ \frac{NdM - MdN}{dx^2}dF + \frac{NNdF}{dx} - \frac{2FdM(NdM - MdN)}{Mdx^2} - \frac{2NNFdM}{Mdx},$$

quae aequatio per $\frac{dx}{MM}$ multiplicata commode integrabilis redditur, inveniturque integrale

$$C = H - \frac{N}{M}G + \frac{NdM - MdN}{MMdx}F + \frac{NNF}{MM}.$$

Quodsi ergo brevitatis gratia ponamus $N = Ms$, erit

$$C = H - Gs - F\frac{ds}{dx} + Fss$$

seu

$$ds + \frac{G}{F}sdx - ssdx + \frac{(C - H)dx}{F} = 0.$$

Sive iam hinc definiatur quantitas $s = \frac{N}{M}$ sive una functionum F, G et H, pro ipsa aequatione proposita litterae P, Q et R ita determinabuntur, ut sit

$$\text{I. } P = F,$$
$$\text{II. } Q = G + \frac{dF}{dx} - \frac{2FdM}{Mdx},$$

et ex ultima aequatione derivatur

$$R = H + \frac{MdH}{Ndx} - \frac{FddN}{Ndx^2} - \frac{dN}{Ndx}\left(G + \frac{dF}{dx} - \frac{2FdM}{Mdx}\right),$$

qui valor ob $N = Ms$ evadit

$$R = H + \frac{dH}{s\,dx} - \frac{Gds}{s\,dx} - \frac{GdM}{Mdx} - \frac{Fdds}{sdx^2} - \frac{FddM}{Mdx^2} + \frac{2FdM^2}{MMdx^2} - \frac{dFds}{sdx^2} - \frac{dFdM}{Mdx^2},$$

et cum aequatio inventa, si differentietur, det

$$0 = dH - Gds - sdG - \frac{Fdds}{dx} - \frac{dFds}{dx} + 2Fsds + ssdF,$$

obtinebimus

$$\text{III.}\quad R = H - \frac{GdM}{Mdx} + \frac{dG}{dx} - \frac{FddM}{Mdx^2} - \frac{2Fds}{dx} + \frac{2FdM^2}{MMdx^2} - \frac{sdF}{dx} - \frac{dFdM}{Mdx^2};$$

unde, si aequatio

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

resolutionem admittat, etiam resolutio succedet huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz,$$

cum sit

$$z = M\left(\frac{dv}{dx}\right) + Nv = M\left(sv + \left(\frac{dv}{dx}\right)\right).$$

COROLLARIUM 1

350. Si ponatur $M = 1$, ut fiat $z = sv + \left(\frac{dv}{dx}\right)$, erit

$$P = F, \quad Q = G + \frac{dF}{dx} \quad \text{et} \quad R = H + \frac{dG}{dx} - \frac{2Fds + sdF}{dx}$$

neque hoc modo usus istius reductionis restringitur, quoniam, si deinceps loco z ponatur Mz, etiam aequationis hinc ortae resolutio est in promtu.

COROLLARIUM 2

351. Quoties ergo aequationis

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

resolutio est in potestate, toties etiam huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = F\left(\frac{ddz}{dx^2}\right) + \left(G + \frac{dF}{dx}\right)\left(\frac{dz}{dx}\right) + \left(H + \frac{dG}{dx} - \frac{2Fds + sdF}{dx}\right)z$$

resolutio succedit, si modo capiatur s ex hac aequatione

$$Fds + Gsdx - Fssdx + (C - H)dx = 0;$$

tum enim erit $z = sv + \left(\frac{dv}{dx}\right)$. Sunt autem litterae F, G, H functiones ipsius x tantum.

SCHOLION

352. Haec reductio methodum maxime naturalem suppeditare videtur eiusmodi integrationes perficiendi, quae simul functionum differentialia involvunt. Si enim aequationis pro v datae integrale sit $v = \varphi : t$ existente t functione ipsarum x et y, ob $dv = dt\,\varphi' : t$ erit $\left(\frac{dv}{dx}\right) = \left(\frac{dt}{dx}\right)\varphi' : t$ [et] aequationis inde derivatae pro z habebimus [integrale]

$$z = s\varphi : t + \left(\frac{dt}{dx}\right)\varphi' : t.$$

Deinde si fuerit generalius $v = u\varphi : t$, fiet

$$z = su\varphi : t + \left(\frac{du}{dx}\right)\varphi : t + u\left(\frac{dt}{dx}\right)\varphi' : t,$$

unde ratio perspicitur ad eiusmodi aequationes perveniendi, quarum integralia praeter functionem $\varphi : t$ etiam functiones ex eius differentiatione natas $\varphi' : t$ atque adeo etiam sequentes $\varphi'' : t$, $\varphi''' : t$ etc. complectantur. Quamobrem operae pretium erit hanc reductionem accuratius evolvere.

PROBLEMA 57

353. *Concessa resolutione huius aequationis*

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) + \frac{m}{x}\left(\frac{dv}{dx}\right) + \frac{n}{xx}v$$

invenire aliam aequationem huius formae

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz,$$

pro qua sit

$$z = sv + \left(\frac{dv}{dx}\right).$$

SOLUTIO

Facta comparatione cum praecedente problemate habemus

$$F = 1, \quad G = \frac{m}{x} \quad \text{et} \quad H = \frac{n}{xx},$$

unde quantitatem s ex hac aequatione definiri oportet

$$ds + \frac{msdx}{x} - ssdx + \left(f - \frac{n}{xx}\right)dx = 0,$$

qua inventa ob $\frac{dG}{dx} = -\frac{m}{xx}$ aequatio quaesita erit

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \left(\frac{n-m}{xx} - \frac{2ds}{dx}\right)z$$

seu loco ds valore inde substituto

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \left(2f - \frac{m+n}{xx} + \frac{2ms}{x} - 2ss\right)z,$$

pro qua est

$$z = sv + \left(\frac{dv}{dx}\right).$$

I. Ponamus primo quantitatem constantem $f = 0$, ut sit

$$ds + \frac{msdx}{x} - ssdx - \frac{ndx}{xx} = 0,$$

cuius integrale particulare est $s = \frac{\alpha}{x}$ existente

$$-\alpha + m\alpha - \alpha\alpha - n = 0 \quad \text{seu} \quad \alpha\alpha - (m-1)\alpha + n = 0,$$

ex quo ob $\frac{ds}{dx} = \frac{-\alpha}{xx}$ oritur haec aequatio

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \frac{2\alpha - m + n}{xx}z,$$

pro qua est

$$z = \frac{\alpha}{x}v + \left(\frac{dv}{dx}\right),$$

seu exclusa $n = \alpha(m - 1 - \alpha)$, si constet resolutio huius aequationis

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) + \frac{m}{x}\left(\frac{dv}{dx}\right) + \frac{\alpha(m-1-\alpha)}{xx}v,$$

pro hac

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \frac{(\alpha-1)(m-\alpha)}{xx}z$$

erit

$$z = \frac{\alpha}{x}v + \left(\frac{dv}{dx}\right).$$

II. Maneat $f = 0$ et quaeramus pro s valorem completum ponendo $s = \frac{\alpha}{x} + \frac{1}{t}$ fietque ob $n = (m-1)\alpha - \alpha\alpha$

$$dt + \frac{(2\alpha - m)t\,dx}{x} + dx = 0,$$

quae per $x^{2\alpha - m}$ multiplicata et integrata praebet

$$t = \frac{cx^{m-2\alpha}}{2\alpha - m + 1} - \frac{x}{2\alpha - m + 1}$$

hincque

$$s = \frac{\alpha c x^{m-2\alpha-1} + \alpha - m + 1}{x(cx^{m-2\alpha-1} - 1)} = \frac{\alpha}{x} + \frac{2\alpha - m + 1}{x(cx^{m-2\alpha-1} - 1)},$$

unde fit

$$\frac{ds}{dx} = \frac{-\alpha}{xx} + \frac{(m-2\alpha-1)(m-2\alpha)}{xx(cx^{m-2\alpha-1}-1)} + \frac{(m-2\alpha-1)^2}{xx(cx^{m-2\alpha-1}-1)^2}.$$

Hic praecipue notetur casus $c = 0$, quo fit

$$s = \frac{m - \alpha - 1}{x} \quad \text{et} \quad \frac{ds}{dx} = \frac{-m + \alpha + 1}{xx},$$

ita ut data aequatione

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dv}{dx}\right) + \frac{\alpha(m-1-\alpha)}{xx}v$$

pro hac aequatione

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \frac{(\alpha+1)(m-2-\alpha)}{xx}z$$

futurum sit

$$z = \frac{m - \alpha - 1}{x}v + \left(\frac{dv}{dx}\right).$$

Pro generali autem valore sit $m-2\alpha-1=\beta$, ut habeatur

$$s=\frac{\alpha}{x}-\frac{\beta}{x(cx^{\beta}-1)} \quad \text{et} \quad \frac{ds}{dx}=\frac{-\alpha}{xx}+\frac{\beta(\beta+1)}{xx(cx^{\beta}-1)}+\frac{\beta\beta}{xx(cx^{\beta}-1)^2},$$

unde, si detur haec aequatio

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{2\alpha+\beta+1}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(\alpha+\beta)}{xx}v,$$

eius ope resolvetur haec

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{2\alpha+\beta+1}{x}\left(\frac{dz}{dx}\right)$$

$$+\left((\alpha-1)(\alpha+\beta+1)-\frac{2\beta(\beta+1)}{cx^{\beta}-1}-\frac{2\beta\beta}{(cx^{\beta}-1)^2}\right)\frac{z}{xx},$$

cum sit

$$z=\left(\alpha-\frac{\beta}{cx^{\beta}-1}\right)\frac{v}{x}+\left(\frac{dv}{dx}\right).$$

III. Rationem quoque habeamus constantis f ponamusque $f=\frac{1}{aa}$, ut facto $n=\alpha(m-1-\alpha)$ habeamus

$$ds+\frac{msdx}{x}-ssdx-\frac{\alpha(m-1-\alpha)dx}{xx}+\frac{dx}{aa}=0,$$

quae posito $s=\frac{\alpha}{x}+\frac{1}{t}$ abit in

$$dt-\frac{(m-2\alpha)tdx}{x}+dx=\frac{tt}{aa}dx.$$

Sit $m-2\alpha=\gamma$, ut aequatio data sit

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{2\alpha+\gamma}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(\alpha+\gamma-1)}{xx}v$$

et inventa quantitate s prodeat haec aequatio

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{2\alpha+\gamma}{x}\left(\frac{dz}{dx}\right)+\left(\frac{\alpha\alpha-3\alpha+\alpha\gamma-\gamma}{xx}-\frac{2ds}{dx}\right)z$$

seu

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{2\alpha+\gamma}{x}\left(\frac{dz}{dx}\right)+\left(\frac{(\alpha-1)(\alpha+\gamma)}{xx}+\frac{2dt}{ttdx}\right)z,$$

pro qua est

$$z = \left(\frac{\alpha}{x} + \frac{1}{t}\right)v + \left(\frac{dv}{dx}\right),$$

ubi totum negotium ad inventionem quantitatis t redit ex aequatione

$$dt - \frac{\gamma t dx}{x} + dx = \frac{tt}{aa}dx.$$

Hunc in finem statuatur $t = a - \frac{aadu}{udx}$ ac reperitur

$$\frac{ddu}{dx^2} - \frac{\gamma du}{xdx} - \frac{2du}{adx} + \frac{\gamma u}{ax} = 0,$$

cuius duplex resolutio datur, altera ponendo

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

existente

$$B = \frac{\gamma}{\gamma a}A, \quad C = \frac{\gamma - 2}{2(\gamma - 1)a}B, \quad D = \frac{\gamma - 4}{3(\gamma - 2)a}C, \quad E = \frac{\gamma - 6}{4(\gamma - 3)a}D \quad \text{etc.},$$

altera vero ponendo

$$u = Ax^{\gamma+1} + Bx^{\gamma+2} + Cx^{\gamma+3} + Dx^{\gamma+4} + Ex^{\gamma+5} + \text{etc.},$$

ubi

$$B = \frac{\gamma + 2}{(\gamma + 2)a}A, \quad C = \frac{\gamma + 4}{2(\gamma + 3)a}B, \quad D = \frac{\gamma + 6}{3(\gamma + 4)a}C, \quad E = \frac{\gamma + 8}{4(\gamma + 5)a}D \quad \text{etc.};$$

quarum illa abrumpitur, si sit γ numerus integer par positivus, haec vero, si negativus. Qui valores etsi sunt particulares, tamen supra[1]) iam ostendimus, quomodo inde valores completi sint eliciendi.

COROLLARIUM 1

354. Supra autem vidimus (§ 333) hanc aequationem

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) + \frac{2m}{x}\left(\frac{dv}{dx}\right) + \frac{(m+i)(m-i-1)}{xx}v$$

1) Vide L. Euleri *Institutionum calculi integralis* vol. II, § 837, *Leonhardi Euleri Opera omnia*, series I, vol. 12, p. 74. Cf. etiam ibidem § 967, p. 177 et § 1036, p. 230. Aequatio enim hic tractata ex aequatione § 967 oritur ponendo $n = 1$, $a = 1$, $b = 0$, $c = -\gamma$, $e = -\frac{2}{a}$, $f = 0$, $g = \frac{\gamma}{a}$, in § 1036 vero aequatio huius formae resolvitur per quadraturas. F. E.

esse integrabilem, si sit i numerus integer quicunque, unde colligimus hanc aequationem

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{m}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(m-1-\alpha)}{xx}v$$

integrationem admittere, quoties fuerit vel $\alpha=\frac{1}{2}m+i$ vel $\alpha=\frac{1}{2}m-i-1$ seu $m-2\alpha$ numerus integer par sive positivus sive negativus, qui casus ob $m-2\alpha=\gamma$ cum casibus integrabilitatis pro valore generali ipsius s inveniendo congruunt.

COROLLARIUM 2

355. Quando autem ex hac aequatione functionem v definire licet, tum etiam hae duae sequentes aequationes [§ 353] illi similes resolvi poterunt

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha-1)(m-\alpha)}{xx}z$$

et

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha+1)(m-\alpha-2)}{xx}z,$$

cum pro illa sit

$$z=\frac{\alpha}{x}v+\left(\frac{dv}{dx}\right),$$

pro hac vero

$$z=\frac{m-\alpha-1}{x}v+\left(\frac{dv}{dx}\right).$$

COROLLARIUM 3

356. Praeterea vero etiam aequationes alius generis, ubi postremus terminus non est formae $\frac{n}{xx}z$, resolvi possunt, quae inveniuntur, si quantitatis s valor generalius investigatur atque adeo constantis f ratio habetur.

EXEMPLUM 1

357. *Proposita aequatione* $\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)$, *pro qua est*

$$v=\pi:(x+y)+\varphi:(x-y),$$

invenire aequationes magis complicatas, quae huius ope integrari queant.

Cum hic sit $F=1$, $G=0$ et $H=0$, resolvatur haec aequatio

$$ds - ssdx + Cdx = 0$$

et huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2ds}{dx}z$$

integrale erit

$$z = sv + \left(\frac{dv}{dx}\right).$$

Sumta autem primo constante $C=0$ fit $\frac{ds}{ss} = dx$ et $\frac{1}{s} = c - x$ seu $s = \frac{1}{c-x}$ atque $\frac{ds}{dx} = \frac{1}{(c-x)^2}$, ubi quidem sine ulla restrictione poni potest $c=0$, ut huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2}{xx}z$$

integrale sit

$$z = -\frac{1}{x}(\pi:(x+y) + \varphi:(x-y)) + \pi':(x+y) + \varphi':(x-y).$$

Sit deinde $C = aa$ et ob $ds = dx(ss-aa)$ fiet $x = \frac{1}{2a}l\frac{s-a}{s+a} - \frac{1}{2a}lA$ hincque

$$\frac{s-a}{s+a} = Ae^{2ax} \quad \text{et} \quad s = \frac{a(1+Ae^{2ax})}{1-Ae^{2ax}},$$

unde

$$\frac{ds}{dx} = \frac{4Aaae^{2ax}}{(1-Ae^{2ax})^2},$$

et aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{8Aaae^{2ax}}{(1-Ae^{2ax})^2}z$$

integrale est

$$z = \frac{a(1+Ae^{2ax})}{1-Ae^{2ax}}v + \left(\frac{dv}{dx}\right).$$

Sit tandem $C = -aa$ et ob $ds = dx(aa+ss)$ fit $ax + b = \text{Ang. tang.}\frac{s}{a}$ hincque

$$s = a\,\text{tang.}(ax+b) \quad \text{et} \quad \frac{ds}{dx} = \frac{aa}{\cos.(ax+b)^2},$$

quocirca huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2aa}{\cos.(ax+b)^2}z$$

integrale est

$$z = \frac{a \sin.(ax+b)}{\cos.(ax+b)} v + \left(\frac{dv}{dx}\right).$$

EXEMPLUM 2

358. *Proposita aequatione*

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) - \frac{2}{xx} v,$$

cuius integrale constat [§ 357], *invenire alias eius ope integrabiles.*

Pro hoc casu habemus

$$ds - ssdx + \left(C + \frac{2}{xx}\right) dx = 0,$$

qua resoluta erit huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - 2\left(\frac{1}{xx} + \frac{ds}{dx}\right) z$$

integrale

$$z = sv + \left(\frac{dv}{dx}\right).$$

I. Sit primo $C = 0$ et ex aequatione $ds - ssdx + \frac{2dx}{xx} = 0$ fit particulariter $s = \frac{1}{x}$ vel $s = -\frac{2}{x}$. Ponatur ergo $s = \frac{1}{x} + \frac{1}{t}$ eritque

$$dt + \frac{2tdx}{x} + dx = 0,$$

hinc $txx + \frac{1}{3}x^3 = \frac{1}{3}a^3$. Ergo

$$t = \frac{a^3 - x^3}{3xx} \quad \text{et} \quad s = \frac{a^3 + 2x^3}{x(a^3 - x^3)} \quad \text{ideoque} \quad \frac{ds}{dx} + \frac{1}{xx} = \frac{3x(2a^3 + x^3)}{(a^3 - x^3)^2},$$

unde huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{6x(2a^3 + x^3)}{(a^3 - x^3)^2} z$$

integrale est

$$z = \frac{a^3 + 2x^3}{x(a^3 - x^3)} v + \left(\frac{dv}{dx}\right).$$

II. Sit $C = \frac{1}{cc}$ et posito $s = \frac{1}{x} + \frac{1}{t}$ fit

$$dt + \frac{2tdx}{x} + dx = \frac{ttdx}{cc},$$

cui particulariter satisfacit $t = c + \frac{cc}{x}$, ut sit

$$s = \frac{cc + cx + xx}{cx(c+x)} \quad \text{et} \quad \frac{ds}{dx} + \frac{1}{xx} = \frac{1}{(c+x)^2}$$

atque huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2}{(c+x)^2}z$$

integrale sit

$$z = \frac{cc + cx + xx}{cx(c+x)}v + \left(\frac{dv}{dx}\right).$$

Ad integrale autem pro t completum inveniendum statuatur

$$t = c + \frac{cc}{x} + \frac{1}{u}$$

fietque

$$du + \frac{2udx}{c} + \frac{dx}{cc} = 0 \quad \text{seu} \quad dx = \frac{-ccdu}{1 + 2cu},$$

hinc

$$x = b - \frac{c}{2}l(1 + 2cu),$$

ergo

$$u = \frac{e^{\frac{2(b-x)}{c}} - 1}{2c},$$

unde

$$t = c + \frac{cc}{x} + \frac{2c}{e^{\frac{2(b-x)}{c}} - 1} \quad \text{et} \quad s = \frac{1}{x} + \frac{x\left(e^{\frac{2(b-x)}{c}} - 1\right)}{c\left((c+x)e^{\frac{2(b-x)}{c}} + x - c\right)}$$

atque

$$\frac{ds}{dx} + \frac{1}{xx} = \frac{-dt}{ttdx} = \frac{1}{tt}\left(1 + \frac{2t}{x} - \frac{tt}{cc}\right) = \frac{1}{tt}\left(\frac{cc}{xx} - \frac{4e^{\frac{2(b-x)}{c}}}{\left(e^{\frac{2(b-x)}{c}} - 1\right)^2}\right).$$

SCHOLION

359. Quoniam supra [§ 333] invenimus hanc aequationem

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) - \frac{i(i+1)}{xx}v$$

integrationem admittere, quippe qui casus oritur ex generali forma (§ 354) sumto $m=0$, erit problemate huc translato

$$ds - ssdx + \left(f + \frac{i(i+1)}{xx}\right)dx = 0$$

hincque inventa quantitate s huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \left(2f + \frac{i(i+1)}{xx} - 2ss\right)z$$

integrale erit

$$z = sv + \left(\frac{dv}{dx}\right).$$

I. Quodsi iam capiamus $f=0$, erit particulariter $s = \frac{i}{x}$ vel $s = \frac{-i-1}{x}$, unde quidem aequationis integrabilis forma non mutatur. At facto $s = \frac{i}{x} + \frac{1}{t}$ oritur

$$dt + \frac{2itdx}{x} + dx = 0,$$

cuius integrale est

$$x^{2i}t + \frac{1}{2i+1}x^{2i+1} = \frac{g}{2i+1},$$

ideoque

$$s = \frac{ig + (i+1)x^{2i+1}}{x(g - x^{2i+1})}$$

et aequatio integrabilis fit

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{i(i-1)gg + 6i(i+1)gx^{2i+1} + (i+1)(i+2)x^{4i+2}}{xx(g - x^{2i+1})^2}z.$$

II. At non reiecto f sit $s = \frac{i}{x} + u$ fietque

$$-du + \frac{2iudx}{x} + uudx = fdx;$$

quae ut in aequationem differentialem secundi gradus facile per seriem resolubilem convertatur, ponatur

$$u = \sqrt{f} - \frac{i}{x} - \frac{dr}{rdx}$$

et prodit[1])

$$\frac{ddr}{dx^2} - \frac{2dr\sqrt{f}}{dx} - \frac{i(i+1)r}{xx} = 0.$$

Sit $\sqrt{f} = a$ et statuatur

$$r = Ax^{i+1} + Bx^{i+2} + Cx^{i+3} + Dx^{i+4} + \text{etc.}$$

ac reperitur

$$B = \frac{2(i+1)a}{1(2i+2)}A, \quad C = \frac{2(i+2)a}{2(2i+3)}B, \quad D = \frac{2(i+3)a}{3(2i+4)}C, \quad E = \frac{2(i+4)a}{4(2i+5)}D \quad \text{etc.},$$

quae abrumpitur, quoties i est numerus integer negativus.

Sin autem statuatur

$$r = Ax^{-i} + Bx^{1-i} + Cx^{2-i} + Dx^{3-i} + \text{etc.},$$

sequens relatio nascitur

$$B = \frac{2ia}{2i}A, \quad C = \frac{2(i-1)a}{2(2i-1)}B, \quad D = \frac{2(i-2)a}{3(2i-2)}C, \quad E = \frac{2(i-3)a}{4(2i-3)}D \quad \text{etc.},$$

quae abrumpitur, quoties i est numerus integer positivus.

PROBLEMA 58

360. *Proposita aequatione*

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) - \frac{2aa}{\cos.(ax+b)^2}v,$$

cuius integrale est [§ 357]

$$v = a\,\text{tang.}(ax+b)\cdot\big(\pi:(x+y) + \varphi:(x-y)\big) + \pi':(x+y) + \varphi':(x-y),$$

per transformationem hic traditam alias invenire aequationes eius ope integrabiles.

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1) Vide notam p. 247; nam et hic habetur casus specialis aequationis tractatae in § 967 voluminis II. F. E.

SOLUTIO

Ponamus brevitatis gratia angulum $ax+b=\omega$, ut sit $d\omega = adx$, et ex § 351, cum sit $F=1$, $G=0$, $H=\frac{-2aa}{\cos.\omega^2}$, quaeratur quantitas s ex hac aequatione

$$ds - ssdx + \left(C + \frac{2aa}{\cos.\omega^2}\right)dx = 0$$

eritque huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \left(\frac{2aa}{\cos.\omega^2} + \frac{2ds}{dx}\right)z$$

integrale $z = sv + \left(\frac{dv}{dx}\right)$ seu

$$z = as\,\mathrm{tang}.\,\omega\cdot(\pi:(x+y)+\varphi:(x-y)) + s(\pi':(x+y)+\varphi':(x-y))$$
$$+\frac{aa}{\cos.\omega^2}(\pi:(x+y)+\varphi:(x-y)) + a\,\mathrm{tang}.\,\omega\cdot(\pi':(x+y)+\varphi':(x-y))$$
$$+\pi'':(x+y)+\varphi'':(x-y).$$

Totum ergo negotium ad inventionem quantitatis s reducitur, quem in finem ponamus

$$s = \alpha\,\mathrm{tang}.\,\omega - \frac{du}{udx},$$

fietque

$$\frac{ds}{dx} = \frac{\alpha a}{\cos.\omega^2} - \frac{ddu}{udx^2} + \frac{du^2}{uudx^2}$$

et facta substitutione prodit

$$\frac{\alpha a}{\cos.\omega^2} - \frac{\alpha\alpha\sin.\omega^2}{\cos.\omega^2} + C + \frac{2aa}{\cos.\omega^2} - \frac{ddu}{udx^2} + \frac{2\alpha du}{udx}\,\mathrm{tang}.\,\omega = 0.$$

Iam ob

$$-\frac{\alpha\alpha\sin.\omega^2}{\cos.\omega^2} = -\frac{\alpha\alpha}{\cos.\omega^2} + \alpha\alpha$$

sumatur α ita, ut fiat

$$-\alpha\alpha + \alpha a + 2aa = 0.$$

Capiatur ergo $\alpha = -a$, ut sit

$$s = -a\,\mathrm{tang}.\,\omega - \frac{du}{udx},$$

et pro quantitate u invenienda haec habetur aequatio

$$\frac{ddu}{u\,dx^2} + \frac{2\,adu}{udx}\,\text{tang.}\,\omega + naau = 0$$

posito $C = -aa - naa$ seu

$$\frac{ddu}{d\omega^2} + \frac{2\,du}{d\omega}\,\text{tang.}\,\omega + nu = 0$$

ob $dx = \frac{d\omega}{a}$; cuius resolutio non parum ardua videtur, inter complures autem modos eam tractandi hic ad institutum maxime idoneus videtur.

Fingatur

$$u = A\cos.\lambda\omega + B\cos.(\lambda+2)\omega + C\cos.(\lambda+4)\omega + \text{etc.}$$

eritque

$$\frac{du}{d\omega} = -\lambda A\sin.\lambda\omega - (\lambda+2)B\sin.(\lambda+2)\omega - (\lambda+4)C\sin.(\lambda+4)\omega - \text{etc.},$$

$$\frac{ddu}{d\omega^2} = -\lambda\lambda A\cos.\lambda\omega - (\lambda+2)^2B\cos.(\lambda+2)\omega - (\lambda+4)^2C\cos.(\lambda+4)\omega - \text{etc.}$$

et aequatio hac forma repraesentata

$$\frac{2\,ddu}{d\omega^2}\cos.\omega + \frac{4\,du}{d\omega}\sin.\omega + 2nu\cos.\omega = 0$$

dabit

$$\begin{array}{llll}
0 = -\lambda\lambda A\cos.(\lambda-1)\omega & -(\lambda+2)^2B\cos.(\lambda+1)\omega & -(\lambda+4)^2C\cos.(\lambda+3)\omega & -\text{etc.}\\
 & -\quad\lambda\lambda A & -(\lambda+2)^2B & \\
-2\lambda A & -2(\lambda+2)B & -2(\lambda+4)C & \\
 & +\quad 2\lambda A & +2(\lambda+2)B & \\
+\quad nA & +\quad nB & +\quad nC & \\
 & +\quad nA & +\quad nB &
\end{array}$$

unde λ ita capi oportet, ut sit

$$\lambda\lambda + 2\lambda = n \quad \text{seu} \quad \lambda = -1 \pm \sqrt{(n+1)}$$

duplexque pro λ habeatur valor. Praeterea vero secundus terminus ob $n = \lambda\lambda + 2\lambda$ praebet $B = \frac{\lambda}{\lambda+2}A$, tertius vero commode dat $C = 0$, unde et sequentes omnes evanescunt.

Sumamus $n = mm - 1$, ut sit

$$\lambda = -1 \pm m \quad \text{et} \quad B = \frac{-1 \pm m}{1 \pm m} A,$$

atque integrale completum concludi videtur

$$u = A\left(\cos.(m-1)\omega + \frac{m-1}{m+1}\cos.(m+1)\omega\right)$$
$$+ \mathfrak{A}\left(\cos.(m+1)\omega + \frac{m+1}{m-1}\cos.(m-1)\omega\right).$$

Sit

$$A = (m+1)B \quad \text{et} \quad \mathfrak{A} = (m-1)\mathfrak{B};$$

fiet

$$u = (m+1)(B+\mathfrak{B})\cos.(m-1)\omega + (m-1)(B+\mathfrak{B})\cos.(m+1)\omega;$$

ubi cum binae constantes in unam coalescant, hoc integrale tantum est particulare, ex quo autem deinceps completum elici poterit [§ 361, 362]. Cum ergo sit

$$\frac{du}{ud\omega} = \frac{-(mm-1)\sin.(m-1)\omega - (mm-1)\sin.(m+1)\omega}{(m+1)\cos.(m-1)\omega + (m-1)\cos.(m+1)\omega},$$

est

$$\frac{s}{a} = -\operatorname{tang}.\omega + \frac{(mm-1)(\sin.(m-1)\omega + \sin.(m+1)\omega)}{(m+1)\cos.(m-1)\omega + (m-1)\cos.(m+1)\omega}$$

pro aequatione

$$\frac{ds}{ad\omega} - \frac{ss}{aa} - mm + \frac{2}{\cos.\omega^2} = 0$$

ob $C = -(n+1)aa = -mmaa$.

Illud autem integrale inventum ad hanc formam reducitur

$$\frac{s}{a} = -\operatorname{tang}.\omega + \frac{(mm-1)\operatorname{tang}.m\omega}{m + \operatorname{tang}.m\omega \operatorname{tang}.\omega},$$

quae expressio substituta illi aequationi egregie satisfacere deprehenditur. Scribamus eius loco Θ ac ponamus $\frac{s}{a} = \Theta + \frac{1}{t}$ pro integrali completo eliciendo prodibitque

$$-\frac{dt}{ttd\omega} - \frac{2\Theta}{t} - \frac{1}{tt} = 0 \quad \text{seu} \quad dt + 2\Theta t d\omega + d\omega = 0.$$

Erat autem modo ante

$$\Theta = \frac{s}{a} = -\operatorname{tang.}\omega - \frac{du}{ud\omega},$$

unde

$$\int \Theta d\omega = l\cos.\omega - lu \quad \text{et} \quad e^{2\int \Theta d\omega} = \frac{\cos.\omega^2}{uu},$$

qui est multiplicator pro illa aequatione, sicque fit

$$\frac{t\cos.\omega^2}{uu} = C - \int \frac{d\omega\cos.\omega^2}{uu}.$$

At est

$$u = 2m\cos.m\omega\cos.\omega + 2\sin.m\omega\sin.\omega$$

ideoque

$$\frac{t}{(m\cos.m\omega + \sin.m\omega\operatorname{tang.}\omega)^2} = A - \int \frac{d\omega}{(m\cos.m\omega + \sin.m\omega\operatorname{tang.}\omega)^2},$$

cuius postremi membri integrale deprehenditur

$$\frac{-m\operatorname{tang.}m\omega + \operatorname{tang.}\omega}{m(mm-1)(m + \operatorname{tang.}m\omega\operatorname{tang.}\omega)} = \frac{-m\sin.m\omega + \operatorname{tang.}\omega\cos.m\omega}{m(mm-1)(m\cos.m\omega + \sin.m\omega\operatorname{tang.}\omega)},$$

ita ut sit

$$\frac{t}{(m\cos.m\omega + \sin.m\omega\operatorname{tang.}\omega)^2} = A + \frac{\cos.m\omega\operatorname{tang.}\omega - m\sin.m\omega}{m(mm-1)(m\cos.m\omega + \sin.m\omega\operatorname{tang.}\omega)}$$

seu

$$\frac{1}{t} = \frac{m(mm-1)}{(C(m\cos.m\omega + \sin.m\omega\operatorname{tang.}\omega) + \cos.m\omega\operatorname{tang.}\omega - m\sin.m\omega)(m\cos.m\omega + \sin.m\omega\operatorname{tang.}\omega)},$$

cui addatur

$$\Theta = -\operatorname{tang.}\omega + \frac{(mm-1)\sin.m\omega}{m\cos.m\omega + \sin.m\omega\operatorname{tang.}\omega},$$

ut prodeat $\frac{s}{a}$, eritque

$$\frac{s}{a} = -\operatorname{tang.}\omega + \frac{(mm-1)(C\sin.m\omega + \cos.m\omega)}{C(m\cos.m\omega + \sin.m\omega\operatorname{tang.}\omega) + \cos.m\omega\operatorname{tang.}\omega - m\sin.m\omega}$$

seu

$$\frac{s}{a} = \frac{(mm-1-\operatorname{tang.}\omega^2)(C\sin.m\omega + \cos.m\omega) - m\operatorname{tang.}\omega(C\cos.m\omega - \sin.m\omega)}{C(m\cos.m\omega + \sin.m\omega\operatorname{tang.}\omega) + \cos.m\omega\operatorname{tang.}\omega - m\sin.m\omega}.$$

COROLLARIUM 1

361. Hic praecipue notandum est huius aequationis

$$\frac{ddu}{d\omega^2} + \frac{2du}{d\omega}\operatorname{tang.}\omega + (mm-1)u = 0$$

integrale particulare esse

$$u = m\cos.m\omega\cos.\omega + \sin.m\omega\sin.\omega;$$

aliud vero integrale particulare reperitur simili modo

$$u = m\sin.m\omega\cos.\omega - \cos.m\omega\sin.\omega,$$

unde concluditur completum

$$u = A(m\cos.m\omega\cos.\omega + \sin.m\omega\sin.\omega) + B(m\sin.m\omega\cos.\omega - \cos.m\omega\sin.\omega).$$

COROLLARIUM 2

362. Si hic ponatur

$$A = C\cos.\alpha \quad \text{et} \quad B = -C\sin.\alpha,$$

hoc integrale completum ad hanc formam redigitur

$$u = C(m\cos.(m\omega+\alpha)\cos.\omega + \sin.(m\omega+\alpha)\sin.\omega),$$

quod quidem ex integrali particulari primum invento statim concludi potuisset, cum ibi loco anguli $m\omega$ scribere liceat $m\omega + \alpha$.

COROLLARIUM 3

363. Hinc multo facilius reperitur valor

$$\frac{s}{a} = -\operatorname{tang.}\omega - \frac{du}{u\,d\omega};$$

cum enim sit

$$\frac{du}{d\omega} = -C(mm-1)\sin.(m\omega+\alpha)\cos.\omega,$$

erit

$$\frac{s}{a} = -\operatorname{tang.}\omega + \frac{(mm-1)\sin.(m\omega+\alpha)\cos.\omega}{m\cos.(m\omega+\alpha)\cos.\omega + \sin.(m\omega+\alpha)\sin.\omega}$$

hincque

$$\frac{ds}{a\,d\omega} = \frac{ds}{aa\,dx} = \frac{-1}{\cos.\omega^2} + \frac{(mm-1)(m^2\cos.\omega^2 - \sin.(m\omega+\alpha)^2)}{(m\cos.(m\omega+\alpha)\cos.\omega + \sin.(m\omega+\alpha)\sin.\omega)^2}$$

et aequatio, cuius integrationem invenimus, erit [§ 360]

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2(mm-1)aa(m^2\cos.\omega^2 - \sin.(m\omega+\alpha)^2)}{(m\cos.(m\omega+\alpha)\cos.\omega + \sin.(m\omega+\alpha)\sin.\omega)^2}\,z$$

eiusque integrale colligitur

$$\begin{aligned}z = {} & \frac{maa(m\sin.(m\omega+\alpha)\sin.\omega+\cos.(m\omega+\alpha)\cos.\omega)}{m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega}(\pi:(x+y)+\varphi:(x-y)) \\ & + \frac{(mm-1)a\sin.(m\omega+\alpha)\cos.\omega}{m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega}(\pi':(x+y)+\varphi':(x-y)) \\ & + (\pi'':(x+y)+\varphi'':(x-y))\end{aligned}$$

existente $\omega = ax + b$.

SCHOLION 1

364. Omnino memoratu digna est integratio huius aequationis

$$\frac{ddu}{d\omega^2} + \frac{2du}{d\omega}\operatorname{tang}.\omega + (mm-1)u = 0,$$

unde occasionem carpo hanc aequationem generaliorem tractandi

$$\frac{ddu}{d\omega^2} + \frac{2fdu}{d\omega}\operatorname{tang}.\omega + gu = 0,$$

quam primum observo posito

$$\frac{du}{u} = -(2f+1)d\omega\operatorname{tang}.\omega + \frac{dv}{v},$$

ut sit

$$u = \cos.\omega^{2f+1}v,$$

abire in hanc formam

$$\frac{ddv}{d\omega^2} - \frac{2(f+1)dv}{d\omega}\operatorname{tang}.\omega + (g-2f-1)v = 0,$$

ita ut, si illa integrabilis existat casu $f = n$, integrabilis quoque sit casu $f = -n-1$.

Iam pro illa aequatione ponatur

$$u = A\sin.\lambda\omega + B\sin.(\lambda+2)\omega + C\sin.(\lambda+4)\omega + D\sin.(\lambda+6)\omega + \text{etc.}$$

et facta substitutione in aequatione

$$\frac{2ddu}{d\omega^2}\cos.\omega + \frac{4fdu}{d\omega}\sin.\omega + 2gu\cos.\omega = 0$$

reperitur

$$\begin{array}{llll}
-\lambda\lambda A\sin.(\lambda-1)\omega & -(\lambda+2)^2 B\sin.(\lambda+1)\omega & -(\lambda+4)^2 C\sin.(\lambda+3)\omega & -(\lambda+6)^2 D\sin.(\lambda+5)\omega \\
 & -\quad \lambda\lambda A & -(\lambda+2)^2 B & -(\lambda+4)^2 C \\
 & +\quad 2\lambda A f & +2(\lambda+2)Bf & +2(\lambda+4)Cf \\
-2\lambda A f & -2(\lambda+2)Bf & -2(\lambda+4)Cf & -2(\lambda+6)Df \\
 & +\quad Ag & +\quad Bg & +\quad Cg \\
+Ag & +\quad Bg & +\quad Cg & +\quad Dg
\end{array}$$

Oportet ergo sit $g = \lambda\lambda + 2\lambda f$; tum vero coefficientes assumti ita determinantur

$$B = \frac{\lambda f}{\lambda + f + 1} A, \quad C = \frac{(\lambda+1)(f-1)}{2(\lambda + f + 2)} B, \quad D = \frac{(\lambda+2)(f-2)}{3(\lambda+f+3)} C \quad \text{etc.}$$

Statuamus ergo $g = mm - ff$, ut fiat $\lambda = m - f$ et aequationes nostrae sint

$$\frac{ddu}{d\omega^2} + \frac{2fdu}{d\omega}\operatorname{tang.}\omega + (mm - ff)u = 0$$

et

$$\frac{ddv}{d\omega^2} - \frac{2(f+1)dv}{d\omega}\operatorname{tang.}\omega + (mm - (f+1)^2)v = 0$$

existente

$$u = v\cos.\omega^{2f+1} \quad \text{seu} \quad v = \frac{u}{\cos.\omega^{2f+1}}.$$

Quoniam nunc series nostra abrumpitur, quoties est f numerus integer, percurramus casus simpliciores.

I. Sit $f = 0$; erit $\lambda = m$ et

$$B = 0, \quad C = 0 \quad \text{etc.}$$

ideoque

$$u = A\sin.m\omega \quad \text{et} \quad v = \frac{A\sin.m\omega}{\cos.\omega}.$$

II. Sit $f = 1$; erit $\lambda = m - 1$ et

$$B = \frac{m-1}{m+1} A, \quad C = 0 \quad \text{etc.},$$

ergo

$$\frac{u}{a} = (m+1)\sin.(m-1)\omega + (m-1)\sin.(m+1)\omega$$

et $v = \frac{u}{\cos.\omega^3}$ seu

$$\frac{u}{2a} = m\sin.m\omega\cos.\omega - \cos.m\omega\sin.\omega.$$

III. Sit $f=2$; erit $\lambda=m-2$ et

$$B=\frac{2(m-2)}{m+1}A,\quad C=\frac{m-1}{2(m+2)}B=\frac{(m-1)(m-2)}{(m+1)(m+2)}A,\quad D=0\quad \text{etc.},$$

hinc

$$\frac{u}{a}=(m+1)(m+2)\sin.(m-2)\omega+2(m-2)(m+2)\sin.m\omega$$
$$+(m-1)(m-2)\sin.(m+2)\omega$$

indeque $v=\frac{u}{\cos.\omega^5}$ seu

$$\frac{u}{2a}=(mm+2)\sin.m\omega\cos.2\omega+(mm-4)\sin.m\omega-3m\cos.m\omega\sin.2\omega.$$

IV. Sit $f=3$; erit $\lambda=m-3$ et

$$B=\frac{3(m-3)}{m+1}A,\quad C=\frac{2(m-2)}{2(m+2)}B\quad\text{et}\quad D=\frac{m-1}{3(m+3)}C,\quad E=0\quad\text{etc.},$$

ergo

$$\frac{u}{a}=(m+1)(m+2)(m+3)\sin.(m-3)\omega+3(m+2)(mm-9)\sin.(m-1)\omega$$
$$+(m-1)(m-2)(m-3)\sin.(m+3)\omega+3(m-2)(mm-9)\sin.(m+1)\omega$$

existente $v=\frac{u}{\cos.\omega^7}$.

V. Sit $f=4$; erit $\lambda=m-4$ ac reperitur

$$\frac{u}{a}=(m+1)(m+2)(m+3)(m+4)\sin.(m-4)\omega$$
$$+4(m+2)(m+3)(mm-16)\sin.(m-2)\omega$$
$$+(m-1)(m-2)(m-3)(m-4)\sin.(m+4)\omega$$
$$+4(m-2)(m-3)(mm-16)\sin.(m+2)\omega+6(mm-9)(mm-16)\sin.m\omega$$

existente $v=\frac{u}{\cos.\omega^9}$, unde ratio progressionis per se est manifesta.

Notari autem convenit, si posuissemus

$$u=A\cos.\lambda\omega+B\cos.(\lambda+2)\omega+C\cos.(\lambda+4)\omega+\text{etc.},$$

easdem coefficientium determinationes prodituras fuisse, ex qua hi duo valores coniuncti integrale completum exhibebunt; quod etiam ex forma inventa colligitur, si modo loco anguli $m\omega$ generalius scribatur $m\omega+\alpha$.

SCHOLION 2

365. Pluribus autem aliis modis eadem aequatio

$$\frac{ddu}{d\omega^2} + \frac{2fdu}{d\omega}\,\mathrm{tang.}\,\omega + gu = 0$$

tractari et eius integrale per series exprimi potest, unde alii casus integrabilitatis obtinentur.

Ad hoc primum notetur posito $u = \sin.\,\omega^\lambda$ fore

$$\frac{du}{d\omega} = \lambda \sin.\,\omega^{\lambda-1} \cos.\,\omega \quad \text{hincque} \quad \frac{du}{d\omega}\,\mathrm{tang.}\,\omega = \lambda \sin.\,\omega^\lambda$$

et

$$\frac{ddu}{d\omega^2} = \lambda(\lambda-1) \sin.\,\omega^{\lambda-2} \cos.\,\omega^2 - \lambda \sin.\,\omega^\lambda = \lambda(\lambda-1)\sin.\,\omega^{\lambda-2} - \lambda\lambda \sin.\,\omega^\lambda.$$

Hinc, si ponamus[1])

$$u = A \sin.\,\omega^\lambda + B \sin.\,\omega^{\lambda+2} + C \sin.\,\omega^{\lambda+4} + D \sin.\,\omega^{\lambda+6} + \text{etc.},$$

facta substitutione adipiscimur

$$\begin{array}{rlll}
0 = \lambda(\lambda-1) A \sin.\,\omega^{\lambda-2} + & (\lambda+2)(\lambda+1) B \sin.\,\omega^\lambda + & (\lambda+4)(\lambda+3) C \sin.\,\omega^{\lambda+2} + \text{etc.,} \\
- & \lambda\lambda A & - (\lambda+2)^2 B \\
+ & 2\lambda f A & + 2(\lambda+2) f B \\
+ & gA & + gB
\end{array}$$

unde sumi oportet vel $\lambda = 0$ vel $\lambda = 1$; tum vero erit

$$B = \frac{\lambda\lambda - 2\lambda f - g}{(\lambda+1)(\lambda+2)} A, \quad C = \frac{(\lambda+2)^2 - 2(\lambda+2)f - g}{(\lambda+3)(\lambda+4)} B \quad \text{etc.}$$

1) Vide notam p. 247. Observandum enim est aequationem differentialem nostram, si ponatur $\sin.\,\omega = x$, hanc induere formam $(1-xx)ddu + (2f-1)xdxdu + gudx^2 = 0$, quae iterum in forma generali § 967 vol. II continetur. Ceterum EULERUS ipse hanc formam tractavit in Commentatione 678 (indicis ENESTROEMIANI): *Methodus nova investigandi omnes casus, quibus hanc aequationem differentialem* $ddy(1-axx) - bxdxdy - cydx^2 = 0$ *resolvere licet*, Institutiones calculi integralis 4, 1794, p. 533; *LEONHARDI EULERI Opera omnia*, series I, vol. 23. F. E.

Hinc duo casus evolvi convenit:

$\lambda = 0$	$\lambda = 1$
$B = \frac{-g}{1\cdot 2}A$	$B = \frac{1-2f-g}{2\cdot 3}A$
$C = \frac{4-4f-g}{3\cdot 4}B$	$C = \frac{9-6f-g}{4\cdot 5}B$
$D = \frac{16-8f-g}{5\cdot 6}C$	$D = \frac{25-10f-g}{6\cdot 7}C$
$E = \frac{36-12f-g}{7\cdot 8}D$	$E = \frac{49-14f-g}{8\cdot 9}D$
etc.	etc.

Integratio ergo succedit, quoties fuerit $g = ii - 2if$ denotante i numerum integrum positivum. Quare cum posito $u = v\cos.\omega^{2f+1}$ aequatio transformata sit

$$\frac{ddv}{d\omega^2} - \frac{2(f+1)dv}{d\omega}\operatorname{tang.}\omega + (g - 2f - 1)v = 0,$$

haec ideoque et illa erit integrabilis, quoties fuerit

$$g = (i+1)^2 + 2(i+1)f,$$

quos binos casus ita uno complecti licet, ut integratio succedat, dum sit

$$g = ii \pm 2if.$$

SCHOLION 3

366. Eidem aequationi adhuc inhaerens, cum posito $u = \cos.\omega^\lambda$ sit

$$\frac{du}{d\omega} = -\lambda\cos.\omega^{\lambda-1}\sin.\omega$$

ideoque

$$\frac{du}{d\omega}\operatorname{tang.}\omega = -\lambda\cos.\omega^{\lambda-2} + \lambda\cos.\omega^\lambda$$

et

$$\frac{ddu}{d\omega^2} = \lambda(\lambda-1)\cos.\omega^{\lambda-2} - \lambda\lambda\cos.\omega^\lambda,$$

statuo

$$u = A\cos.\omega^\lambda + B\cos.\omega^{\lambda+2} + C\cos.\omega^{\lambda+4} + D\cos.\omega^{\lambda+6} + \text{etc.}$$

et facta substitutione orietur

$$\begin{array}{rcrcrl}
0 = \lambda(\lambda-1)A\cos.\omega^{\lambda-2} & + & (\lambda+2)(\lambda+1)B\cos.\omega^{\lambda} & + & (\lambda+4)(\lambda+3)\,C\cos.\omega^{\lambda+2} & + \text{ etc.} \\
 & - & \lambda\lambda A & - & (\lambda+2)^2 B & \\
-2\lambda f A & - & 2(\lambda+2)fB & - & 2(\lambda+4)fC & \\
 & + & 2\lambda f A & + & 2(\lambda+2)fB & \\
 & + & gA & + & gB &
\end{array}$$

Oportet ergo sit vel $\lambda = 0$ vel $\lambda = 2f+1$; tum vero

$$B = \frac{\lambda\lambda - 2\lambda f - g}{(\lambda+2)(\lambda+1-2f)}A, \quad C = \frac{(\lambda+2)^2 - 2(\lambda+2)f - g}{(\lambda+4)(\lambda+3-2f)}B \quad \text{etc.}$$

et ambo casus ita se habebunt:

$$\begin{array}{c|c}
\lambda = 0 & \lambda = 2f+1 \\
B = \dfrac{-g}{2(1-2f)}A & B = \dfrac{1+2f-g}{2(2f+3)}A \\
C = \dfrac{4-4f-g}{4(3-2f)}B & C = \dfrac{9+6f-g}{4(2f+5)}B \\
D = \dfrac{16-8f-g}{6(5-2f)}C & D = \dfrac{25+10f-g}{6(2f+7)}C \\
\text{etc.} & \text{etc.}
\end{array}$$

Ex priori integratio succedit, si

$$g = 4ii - 4if,$$

ex posteriori, si

$$g = (2i+1)^2 + 2(2i+1)f,$$

qui casus cum iis, qui ex transformata nascuntur, iuncti eodem redeunt ac in paragrapho praecedente inventi.[1])

Omnes ergo hactenus inventi integrabilitatis casus huc revocantur, ut posito $g = mm - ff$ sit vel $f = \pm i$ vel $m = i \pm f$, hoc est vel $f = \pm i$ vel

---

1) Notandum est hic ex aequatione transformata novos casus integrabilitatis non nasci, sed eos solos, qui modo inventi sunt; revera ergo § 366 non omnes praebet casus integrabilitatis, qui in § 365 obtinentur. F. E.

$f = \pm i \pm m$. Ceterum hi posteriores casus etiam ex prima resolutione (§ 364) sequuntur, ubi series quoque abrumpitur, si $\lambda = -i$ ideoque

$$g = mm - ff = ii - 2if, \quad \text{ergo} \quad i - f = \pm m$$

et transformatione in subsidium vocata $f = \pm i \pm m$. Contra vero casus primo inventi in resolutionibus posterioribus non occurrunt.

PROBLEMA 59

367. *Concessa huius aequationis integratione*

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

invenire aequationem huius formae

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz,$$

pro qua sit

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv,$$

ubi F, G, H, P, Q, R et r, s sunt functiones ipsius x tantum.

SOLUTIO

Cum sit

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{d^4v}{dx^2dy^2}\right) + r\left(\frac{d^3v}{dx\,dy^2}\right) + s\left(\frac{ddv}{dy^2}\right),$$

ob

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

erit

$$\left(\frac{d^3v}{dx\,dy^2}\right) = F\left(\frac{d^3v}{dx^3}\right) + \begin{matrix}\dfrac{dF}{dx}\\ +G\end{matrix}\left(\frac{ddv}{dx^2}\right) + \begin{matrix}\dfrac{dG}{dx}\\ +H\end{matrix}\left(\frac{dv}{dx}\right) + \frac{dH}{dx}v$$

et

$$\left(\frac{d^4v}{dx^2dy^2}\right) = F\left(\frac{d^4v}{dx^4}\right) + \begin{matrix}\dfrac{2dF}{dx}\\ +G\end{matrix}\left(\frac{d^3v}{dx^3}\right) + \begin{matrix}\dfrac{ddF}{dx^2}\\ +\dfrac{2dG}{dx}\\ +H\end{matrix}\left(\frac{ddv}{dx^2}\right) + \begin{matrix}\dfrac{ddG}{dx^2}\\ +\dfrac{2dH}{dx}\end{matrix}\left(\frac{dv}{dx}\right) + \frac{ddH}{dx^2}v$$

Deinde vero ob

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv$$

fit

$$\begin{aligned}\left(\frac{dz}{dx}\right) = \left(\frac{d^3v}{dx^3}\right) + r\left(\frac{ddv}{dx^2}\right) &+ \frac{dr}{dx}\left(\frac{dv}{dx}\right) + \frac{ds}{dx}v \\ &+ s\end{aligned}$$

et

$$\begin{aligned}\left(\frac{ddz}{dx^2}\right) = \left(\frac{d^4v}{dx^4}\right) + r\left(\frac{d^3v}{dx^3}\right) &+ \frac{2\,dr}{dx}\left(\frac{ddv}{dx^2}\right) &+ \frac{ddr}{dx^2}\left(\frac{dv}{dx}\right) + \frac{dds}{dx^2}v \\ &+ s &+ \frac{2\,ds}{dx}\end{aligned}$$

His iam substitutis necesse est, ut omnes termini affecti per $\left(\frac{d^4v}{dx^4}\right)$, $\left(\frac{d^3v}{dx^3}\right)$, $\left(\frac{ddv}{dx^2}\right)$, $\left(\frac{dv}{dx}\right)$ et v seorsim evanescant, unde sequentes resultant aequationes:

ex	
$\left(\frac{d^4v}{dx^4}\right)$	I. $F = P,$
$\left(\frac{d^3v}{dx^3}\right)$	II. $G + \frac{2\,dF}{dx} + Fr = Pr + Q,$
$\left(\frac{ddv}{dx^2}\right)$	III. $H + \frac{2\,dG}{dx} + \frac{ddF}{dx^2} + Gr + \frac{rdF}{dx} + Fs = P\left(s + \frac{2\,dr}{dx}\right) + Qr + R,$
$\left(\frac{dv}{dx}\right)$	IV. $\frac{2\,dH}{dx} + \frac{ddG}{dx^2} + Hr + \frac{rdG}{dx} + Gs = P\left(\frac{2\,ds}{dx} + \frac{ddr}{dx^2}\right) + Q\left(s + \frac{dr}{dx}\right) + Rr,$
v	V. $\frac{ddH}{dx^2} + \frac{rdH}{dx} + Hs = P\frac{dds}{dx^2} + Q\frac{ds}{dx} + Rs.$

Ex prima fit $P = F$, ex secunda $Q = G + \frac{2\,dF}{dx}$, ex tertia

$$R = H + \frac{2\,dG}{dx} + \frac{ddF}{dx^2} - \frac{rdF + 2\,Fdr}{dx},$$

qui valores in binis ultimis substituti praebent

$$\begin{aligned}\frac{2\,dH}{dx} + \frac{ddG}{dx^2} - \frac{rdG + Gdr}{dx} - \frac{rddF}{dx^2} - \frac{2\,dFdr}{dx^2} - \frac{2\,sdF + 2\,Fds}{dx} \\ + \frac{rrdF + 2\,Frdr}{dx} - \frac{Fddr}{dx^2} = 0\end{aligned}$$

et

$$\frac{ddH}{dx^2} + \frac{rdH}{dx} - \frac{sddF + 2\,dFds + Fdds}{dx^2} - \frac{2\,sdG + Gds}{dx} + \frac{s(rdF + 2\,Fdr)}{dx} = 0,$$

quarum illa sponte est integrabilis praebens

$$2H + \frac{dG}{dx} - Gr - \frac{rdF + Fdr}{dx} - 2Fs + Frr = A,$$

deinde binis illis aequationibus ita repraesentatis

$$-\frac{dd.Fr}{dx^2} - \frac{2d.Fs}{dx} + \frac{d.Frr}{dx} + \frac{ddG}{dx^2} - \frac{d.Gr}{dx} + \frac{2dH}{dx} = 0,$$

$$-\frac{dd.Fs}{dx^2} + \frac{s}{r}\cdot\frac{d.Frr}{dx} - \frac{2sdG + Gds}{dx} + \frac{rdH}{dx} + \frac{ddH}{dx^2} = 0$$

vel adeo hoc modo

$$\frac{dd.(G-Fr)}{dx} - d.r(G - Fr) + 2d.(H - Fs) = 0,$$

$$\frac{dd.(H-Fs)}{dx} + 2Fsdr + rsdF - Gds - 2sdG + rdH = 0$$

ultima vero ita repraesentari potest

$$\frac{dd.(H-Fs)}{dx} - 2sd.(G - Fr) - ds(G - Fr) + rd.(H - Fs) = 0.$$

Quodsi iam prior per $H - Fs$, haec vero per $-(G - Fr)$ multiplicetur, summa fit

$$\begin{aligned}\frac{(H-Fs)dd.(G-Fr) - (G-Fr)dd.(H-Fs)}{dx} &- (G - Fr)(H - Fs)dr\\ + 2(H - Fs)d.(H - Fs) \qquad &- r(H - Fs)d.(G - Fr)\\ + 2s(G - Fr)d.(G - Fr) + (G - Fr)^2 ds &- r(G - Fr)d.(H - Fs) = 0,\end{aligned}$$

cuius integrale manifesto est

$$\frac{(H-Fs)d(G-Fr) - (G-Fr)d(H-Fs)}{dx}$$
$$+ (H - Fs)^2 + (G - Fr)^2 s - (G - Fr)(H - Fs)r = B.$$

Integrale autem prius inventum est

$$\frac{d.(G-Fr)}{dx} - (G - Fr)r + 2(H - Fs) = A,$$

quae per $H - Fs$ multiplicata et ab illa subtracta relinquit

$$-\frac{(G-Fr)\,d.(H-Fs)}{dx} - (H-Fs)^2 + (G-Fr)^2 s = B - A(H-Fs),$$

sicque habentur duae aequationes simpliciter differentiales, ex quibus binas quantitates r et s definiri oportet, quibus cognitis etiam functiones P, Q et R innotescunt.

COROLLARIUM 1

368. Si sit $F = 1$, $G = 0$ et $H = 0$, aequationes inventae erunt

$$-\frac{dr}{dx} + rr - 2s = a \quad \text{et} \quad \frac{sdr - rds}{dx} + ss = b,$$

unde dx eliminando fit

$$\frac{rds - sdr}{dr} = \frac{b - ss}{a + 2s - rr} \quad \text{seu} \quad \frac{rds}{dr} = \frac{b + as + ss - rrs}{a + 2s - rr},$$

cuius resolutio in genere vix suscipienda videtur. Sumtis autem constantibus $a = 0$ et $b = 0$ aequatio $\frac{rds}{dr} = \frac{ss - rrs}{2s - rr}$ posito $s = rrt$ transit in

$$\frac{rdt + 2tdr}{dr} = \frac{tt - t}{2t - 1} \quad \text{seu} \quad \frac{rdt}{dr} = \frac{-3tt + t}{2t - 1},$$

unde fit

$$\frac{dr}{r} = \frac{dt(1-2t)}{t(3t-1)} = \frac{-dt}{t} + \frac{dt}{3t-1} \quad \text{et} \quad r = \frac{\alpha\sqrt[3]{(3t-1)}}{t},$$

hinc

$$s = \frac{\alpha\alpha\sqrt[3]{(3t-1)^2}}{t}.$$

COROLLARIUM 2

369. Pro eodem casu singulari ponamus $3t - 1 = u^3$, ut fiat

$$r = \frac{3\alpha u}{1 + u^3} \quad \text{et} \quad s = \frac{3\alpha\alpha uu}{1 + u^3}.$$

Iam ob $a = 0$ est

$$dx = \frac{dr}{rr - 2s} = \frac{dr}{rr(1 - 2t)} = \frac{3dr}{rr(1 - 2u^3)},$$

at

$$\frac{dr}{rr} = \frac{du}{3\alpha uu} - \frac{2udu}{3\alpha} = \frac{du(1 - 2u^3)}{3\alpha uu},$$

ita ut sit $dx = \frac{du}{\alpha uu}$ hincque

$$\frac{1}{u} = \beta - \alpha x \quad \text{et} \quad u = \frac{1}{\beta - \alpha x},$$

ubi quidem salva generalitate sumi potest $\beta = 0$ et $u = \frac{-1}{\alpha x}$, unde fit

$$r = \frac{-3xx}{x^3 + c^3}$$

facto $\alpha = -\frac{1}{c}$ et

$$s = \frac{3x}{x^3 + c^3}.$$

Tandem ergo colligitur

$$P = 1, \quad Q = 0 \quad \text{et} \quad R = -\frac{2dr}{dx} = \frac{6x(2c^3 - x^3)}{(c^3 + x^3)^2}.$$

COROLLARIUM 3

370. Proposita ergo aequatione $\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right)$, cuius integrale est

$$v = \Gamma : (x + y) + \Delta : (x - y),$$

huius aequationis integrale assignari poterit

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{6x(2c^3 - x^3)}{(c^3 + x^3)^2} z;$$

est enim

$$z = \left(\frac{ddv}{dx^2}\right) - \frac{3xx}{c^3 + x^3}\left(\frac{dv}{dx}\right) + \frac{3x}{c^3 + x^3} v.$$

SCHOLION 1

371. Haec pro casu $F = 1$, $G = 0$ et $H = 0$ multo facilius atque generalius computari possunt pro quocunque valore quantitatis a, dum sit $b = 0$; tum enim altera aequatio statim dat

$$dx = \frac{rds - sdr}{ss}$$

hincque

$$x = \frac{-r}{s} \quad \text{et} \quad s = \frac{-r}{x},$$

ex quo prima aequatio hanc induit formam

$$\frac{dr}{dx} - rr - \frac{2r}{x} + a = 0.$$

Ponamus $r = \frac{a}{t}$; fiet

$$dt + \frac{2tdx}{x} - ttdx + adx = 0,$$

cui particulariter satisfacit

$$t = \sqrt{a} + \frac{1}{x}.$$

Statuatur ergo

$$t = \sqrt{a} + \frac{1}{x} + \frac{1}{u}$$

ac prodit

$$du + dx + 2udx\sqrt{a} = 0,$$

quae per $e^{2x\sqrt{a}}$ multiplicata et integrata praebet

$$e^{2x\sqrt{a}}u + \frac{1}{2\sqrt{a}}e^{2x\sqrt{a}} = \frac{n}{2\sqrt{a}}$$

ideoque

$$\frac{1}{u} = \frac{2e^{2x\sqrt{a}}\sqrt{a}}{n - e^{2x\sqrt{a}}} = \frac{2\sqrt{a}}{ne^{-2x\sqrt{a}} - 1}$$

et

$$t = \frac{1}{x} + \frac{ne^{-2x\sqrt{a}} + 1}{ne^{-2x\sqrt{a}} - 1}\sqrt{a} = \frac{1}{x} + \frac{n + e^{2x\sqrt{a}}}{n - e^{2x\sqrt{a}}}\sqrt{a}$$

et

$$r = \frac{ax(n - e^{2x\sqrt{a}})}{n(x\sqrt{a} + 1) + e^{2x\sqrt{a}}(x\sqrt{a} - 1)}$$

ac propterea

$$s = \frac{-a(n - e^{2x\sqrt{a}})}{n(x\sqrt{a} + 1) + e^{2x\sqrt{a}}(x\sqrt{a} - 1)},$$

tum vero postremo

$$P = 1, \quad Q = 0 \quad \text{et} \quad R = -\frac{2dr}{dx} = -2rr - \frac{4r}{x} + 2a$$

seu

$$R = \frac{-2a(nn - 4naxxe^{2x\sqrt{a}} - 2ne^{2x\sqrt{a}} + e^{4x\sqrt{a}})}{(n(x\sqrt{a} + 1) + e^{2x\sqrt{a}}(x\sqrt{a} - 1))^2} = \frac{-2a(n - e^{2x\sqrt{a}})^2 + 8naaxxe^{2x\sqrt{a}}}{(n(x\sqrt{a} + 1) + e^{2x\sqrt{a}}(x\sqrt{a} - 1))^2}.$$

Si iam sumatur a evanescens et $n = 1 + \frac{2}{3} ac^3 \sqrt{a}$, formulae ante [§ 369] inventae resultant. At si a sit quantitas negativa, puta $a = -m^2$, capiaturque $n = \frac{\alpha\sqrt{-1}+\beta}{\alpha\sqrt{-1}-\beta}$, reperitur

$$r = \frac{-mmx(\beta\cos.mx + \alpha\sin.mx)}{\beta\cos.mx + \alpha\sin.mx - mx(\alpha\cos.mx - \beta\sin.mx)} = \frac{-mmx\cos.(mx+\gamma)}{\cos.(mx+\gamma) + mx\sin.(mx+\gamma)}$$

et

$$s = \frac{mm\cos.(mx+\gamma)}{\cos.(mx+\gamma) + mx\sin.(mx+\gamma)}$$

indeque

$$R = \frac{2mm(\cos.(mx+\gamma)^2 - mmxx)}{(\cos.(mx+\gamma) + mx\sin.(mx+\gamma))^2}.$$

Quantitas R reducitur ad hanc

$$R = \frac{8naaxx - 2a(ne^{-x\sqrt{a}} - e^{x\sqrt{a}})^2}{(n(1+x\sqrt{a})e^{-x\sqrt{a}} - (1-x\sqrt{a})e^{x\sqrt{a}})^2},$$

quae forma sumto a valde parvo abit in

$$R = \frac{8naaxx - 2a\left(n - 1 - (n+1)x\sqrt{a} + \frac{n-1}{2}axx - \frac{n+1}{6}ax^3\sqrt{a} + \text{etc.}\right)^2}{\left(n - 1 - \frac{1}{2}(n-1)axx + \frac{1}{3}(n+1)ax^3\sqrt{a} + \text{etc.}\right)^2}.$$

Statuatur $n = 1 + \beta a\sqrt{a}$, ut sit

$$n - 1 = \beta a\sqrt{a} \quad \text{et} \quad n + 1 = 2 + \beta a\sqrt{a};$$

erit

$$R = \frac{8naaxx - 2a\left(\beta a\sqrt{a} - 2x\sqrt{a} - \beta aax + \frac{\beta aaxx\sqrt{a}}{2} - \frac{1}{3}ax^3\sqrt{a}\right)^2}{\left(\beta a\sqrt{a} - \frac{1}{2}\beta aaxx\sqrt{a} + \frac{2}{3}ax^3\sqrt{a}\right)^2},$$

ubi numerator fit

$$8aaxx + 8\beta a^3xx\sqrt{a} - 2a\left(\beta\beta a^3 - 4\beta aax - 2\beta\beta a^3x\sqrt{a} + 4axx + \frac{4}{3}aax^4\right);$$

ubi cum termini per aa affecti se destruant, retineantur ii soli, qui per a^3 sunt affecti; erit idem in denominatore observando

$$R = \frac{8\beta a^3x - \frac{8}{3}a^3x^4}{a^3(\beta + \frac{2}{3}x^3)^2} = \frac{8x(\beta - \frac{1}{3}x^3)}{(\beta + \frac{2}{3}x^3)^2},$$

quae iam facile ad formam

$$R = \frac{6x(2c^3 - x^3)}{(c^3 + x^3)^2}$$

reducitur sumendo $3\beta = 2c^3$, ut sit $\beta = \frac{2}{3}c^3$. Quare hic casus oritur sumendo a evanescens et $n = 1 + \frac{2}{3}c^3 a\sqrt{a}$.

SCHOLION 2

372. Cum evolutio solutionis inventae sit difficillima neque ulla via pateat, quomodo ambae quantitates incognitae r et s ex binis aequationibus erutis definiri queant, in scientiae incrementum haud parum iuvabit observasse idem problema per repetitionem transformationis in primo problemate [§ 349] huius capitis quoque solvi posse neque proinde usu carebit has duas solutiones inter se comparasse.

Proposita ergo aequatione

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

ponamus primo

$$u = \left(\frac{dv}{dx}\right) + pv$$

ac p ex hac aequatione determinetur [§ 351]

$$Fdp + Gpdx - Fppdx + (C - H)dx = 0$$

ac tum ista resultabit aequatio

$$\left(\frac{ddu}{dy^2}\right) = F\left(\frac{ddu}{dx^2}\right) + \left(G + \frac{dF}{dx}\right)\left(\frac{du}{dx}\right) + \left(H + \frac{dG}{dx} - \frac{2Fdp + pdF}{dx}\right)u.$$

Nunc pro hac aequatione porro transformanda statuamus simili modo

$$z = \left(\frac{du}{dx}\right) + qu,$$

ita ut sit quoque

$$z = \left(\frac{ddv}{dx^2}\right) + (p + q)\left(\frac{dv}{dx}\right) + \left(\frac{dp}{dx} + pq\right)v,$$

et quantitate q ex hac aequatione definita

$$Fdq + \left(G + \frac{dF}{dx}\right)qdx - Fqqdx + \left(D - H - \frac{dG}{dx} + \frac{2Fdp + pdF}{dx}\right)dx = 0$$

orietur haec aequatio

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz,$$

cuius quantitates P, Q, R ita se habent

$$P = F, \quad Q = G + \frac{2dF}{dx}$$

et

$$R = H + \frac{2dG}{dx} - \frac{2Fdp + pdF}{dx} + \frac{ddF}{dx^2} - \frac{2Fdq + qdF}{dx}.$$

Cum hac ergo solutione convenire debet ea, quam postremum problema [§ 367] suppeditavit; in quo cum statim posuerimus

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv,$$

erit utique

$$r = p + q \quad \text{et} \quad s = \frac{dp}{dx} + pq,$$

unde quidem statim valores pro P, Q et R manifesto prodeunt iidem.

Verum multo minus apparet, si pro r et s isti valores p et q substituantur, tum istas binas aequationes

$$\frac{d.(G - Fr)}{dx} - (G - Fr)r + 2(H - Fs) = A$$

et

$$\frac{(G - Fr)d.(H - Fs)}{dx} + (H - Fs)^2 - (G - Fr)^2 s - A(H - Fs) = B$$

ad eas, quas ante invenimus, reduci

$$\frac{Fdp}{dx} + Gp - Fpp - H + C = 0$$

et

$$\frac{Fdq}{dx} + \left(G + \frac{dF}{dx}\right)q - Fqq - H - \frac{dG}{dx} + \frac{2Fdp + pdF}{dx} + D = 0,$$

ita ut hae constantes C et D ad illas A et B certam teneant relationem. Interim patet has postremas aequationes multo esse simpliciores, dum prior duas tantum variabiles p et x complectitur indeque p per x, cuius F, G et H sunt functiones datae, determinari debet, qua inventa quantitatem q simili modo ex altera aequatione elici oportet. Verum in ambabus superioribus aequationibus binae variabiles r et s ita inter se sunt permixtae, ut nulla methodus eas resolvendi vel adeo ad aequationem inter duas tantum variabiles perveniendi habeatur. Cum igitur certum sit priores solutu difficillimas ad posteriores multo faciliores ope substitutionum assignatarum perduci posse, sine dubio methodus hanc reductionem efficiendi haud contemnenda subsidia in Analysin esse allatura videtur.

SCHOLION 3

373. Cum adeo consensus harum duarum solutionum maxime sit absconditus, casum specialem accuratius perpendi expediet.

Sit igitur $F = 1$, $G = 0$ et $H = 0$ ac binae priores aequationes inter r et s has induent formas

$$\text{I. } \frac{-dr}{dx} + rr - 2s = A \quad \text{et} \quad \text{II. } \frac{rds}{dx} + ss - rrs + As = B,$$

posteriores vero istas

$$\text{III. } \frac{dp}{dx} - pp + C = 0 \quad \text{et} \quad \text{IV. } \frac{dq}{dx} - qq + \frac{2dp}{dx} + D = 0,$$

quas cum illis certum est ita cohaerere, ut sit

$$r = p + q \quad \text{et} \quad s = \frac{dp}{dx} + pq.$$

Ut saltem consensum a posteriori agnoscamus, sit $C = -mm$ et tertia dat

$$dx = \frac{dp}{mm + pp},$$

hinc

$$x = \frac{1}{m} \text{ Ang. tang. } \frac{p}{m} \quad \text{et} \quad p = m \text{ tang. } mx.$$

Hinc cum sit $\frac{dp}{dx} = mm + pp$, erit

$$s = mm + pp + pq = mm + pr = m(m + r\,\text{tang.}\,mx),$$

qui valor in I substitutus dat

$$\frac{-dr}{dx} + rr - 2mr\,\text{tang.}\,mx - 2mm = A$$

seu

$$\frac{dr}{dx} = rr - 2mr\,\text{tang.}\,mx - 2mm - A;$$

secunda vero ob

$$\frac{ds}{dx} = \frac{m\,dr}{dx}\,\text{tang.}\,mx + \frac{mmr}{\cos.\,mx^2}$$

abit in

$$\frac{mr\,dr}{dx}\,\text{tang.}\,mx$$

$$= mr^3\,\text{tang.}\,mx - 2mmrr\,\text{tang.}\,mx^2 - m(A + 2mm)r\,\text{tang.}\,mx - m^4 - Amm + B,$$

ex quibus dr eliminando fit $B = Amm + m^4$. Pro quarta vero ob

$$q = r - p = r - m\,\text{tang.}\,mx$$

resultat

$$\frac{dr}{dx} = rr - 2mr\,\text{tang.}\,mx - mm - D,$$

ita ut sit $D = mm + A$. Consensus ergo nostrarum aequationum in hac constantium relatione consistit, ut ob $mm = -C$ sit

$$D = A - C \quad \text{et} \quad B = -C(A - C) = -CD.$$

In genere vero etiam eaedem relationes locum habent; nam si III et IV in unam summam colligantur, ob $C + D = A$ et $p + q = r$ erit

$$\frac{Fdr}{dx} + Gr + \frac{rdF}{dx} - Fpp - Fqq - 2H - \frac{dG}{dx} + \frac{2Fdp}{dx} + A = 0;$$

cum vero sit $\frac{dp}{dx} = s - pq$, fit

$$\frac{Fdr + rdF - dG}{dx} + Gr - Frr - 2H + 2Fs + A = 0$$

seu

$$\frac{d.(G - Fr)}{dx} - (G - Fr)r + 2(H - Fs) = A,$$

quae est ipsa aequatio prima.

Porro aequatio tertia ob $\frac{dp}{dx} = s - pq$ dat

$$Fs - Fpr + Gp - H + C = 0 \quad \text{seu} \quad C = H - Fs - p(G - Fr);$$

quarta vero reducitur ad hanc formam

$$\frac{Fdr}{dx} + Gq + \frac{qdF}{dx} - Fqq - H - \frac{dG}{dx} + Fs - Fpq + \frac{pdF}{dx} + D = 0$$

seu

$$\frac{d.(Fr - G)}{dx} + q(G - Fr) - H + Fs + D = 0$$

hincque

$$D = \frac{d.(G - Fr)}{dx} - q(G - Fr) + H - Fs,$$

ex quibus concluditur

$$CD = \frac{(H - Fs)\,d.(G - Fr)}{dx} - q(G - Fr)(H - Fs) + (H - Fs)^2$$
$$- \frac{p(G - Fr)\,d.(G - Fr)}{dx} + pq(G - Fr)^2 - p(G - Fr)(H - Fs).$$

Ex secunda vero habemus

$$B = \frac{(G - Fr)\,d.(H - Fs)}{dx} - \frac{(H - Fs)\,d.(G - Fr)}{dx}$$
$$- (H - Fs)^2 + (G - Fr)(H - Fs)r - (G - Fr)^2 s,$$

quibus expressionibus coniunctis fit

$$\frac{CD + B}{G - Fr} = \frac{d.(H - Fs)}{dx} - \frac{p\,d.(G - Fr)}{dx} - \frac{dp\,(G - Fr)}{dx}$$
$$= \frac{d.(H - Fs) - d.p(G - Fr)}{dx} = 0,$$

siquidem est $C = H - Fs - p(G - Fr)$, ex quo etiam in genere est

$$B = -CD \quad \text{et} \quad A = C + D.$$

Interim tamen hinc non perspicitur, quomodo ex aequationibus I et II binae reliquae III et IV derivari queant.

SCHOLION 4

374. Omnibus his diligenter pensitatis manifestum fiet totum negotium ope substitutionis satis simplicis confici posse. Quod quo facilius ostendatur, ponamus brevitatis causa $G - Fr = R$ et $H - Fs = S$, ut habeantur hae duae aequationes

$$\text{I.}\quad A = \frac{dR}{dx} - \frac{GR}{F} + \frac{RR}{F} + 2S,$$

$$\text{II.}\quad B = \frac{RdS - SdR}{dx} - \frac{HRR}{F} + \frac{GRS}{F} - SS,$$

ex quibus duas quantitates R et S erui oporteat, dum F, G, H sunt functiones quaecunque ipsius x, at A et B quantitates constantes. Ad hoc adhibeatur ista substitutio $S = C + Rp$ ita adornanda, ut binae illae aequationes coalescant in unam, in qua praeter x unica insit nova variabilis p deinceps per methodos cognitas investiganda.

Hinc ob $dS = Rdp + pdR$ habebitur

$$\text{I.}\quad A = \frac{dR}{dx} - \frac{GR}{F} + \frac{RR}{F} + 2C + 2Rp,$$

$$\text{II.}\quad B = \frac{RRdp}{dx} - \frac{CdR}{dx} - \frac{HRR}{F} + \frac{CGR}{F} + \frac{GRRp}{F} - CC - 2CRp - RRpp,$$

unde primo eliminando dR concluditur

$$B + AC = \frac{RRdp}{dx} + \frac{CRR}{F} + CC - \frac{HRR}{F} - RRpp + \frac{GRRp}{F};\text{[1]}$$

dummodo ergo constantem C ita assumamus, ut sit $CC = B + AC$, per divisionem etiam ipsa quantitas R tolletur resultabitque haec aequatio

$$0 = \frac{dp}{dx} + \frac{C}{F} - \frac{H}{F} - pp + \frac{Gp}{F},$$

cuius resolutio ad methodos magis cognitas pertinet.

Cum igitur ista methodus maximi sit momenti, sequens problema, etiamsi ad primam partem calculi integralis sit referendum, hic adiicere operae pretium videtur.

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1) In editione principe terminus $\frac{GRRp}{F}$ deest itemque postea $\frac{Gp}{F}$. Correxit F. E.

PROBLEMA 60

375. *Propositis huiusmodi duabus aequationibus differentialibus*

$$\text{I.}\quad 0 = \frac{dy}{dx} + F + Gy + Hz + Iyy + Kyz + Lzz,$$

$$\text{II.}\quad 0 = \frac{ydz - zdy}{dx} + P + Qy + Rz + Syy + Tyz + Vzz,$$

ubi F, G, H etc., P, Q, R etc. sint functiones ipsius x, methodum exponere has aequationes, siquidem fieri licet, resolvendi.

SOLUTIO

Methodus indicata in hoc consistit, ut ope substitutionis $z = a + yv$ ex illis aequationibus una elici queat duas tantum variabiles x et v implicans. Quoniam igitur est $ydz - zdy = yydv - ady$, ex $\text{I}a + \text{II}$ nascitur haec aequatio

$$\begin{aligned} 0 = \frac{yydv}{dx} &+ P + Qy + Rz + Syy + Tyz + Vzz \\ &+ aF + aGy + aHz + aIyy + aKyz + aLzz, \end{aligned}$$

quae loco z substituto valore $a + yv$ ita exhibeatur secundum potestates ipsius y

$$\begin{aligned} 0 = \frac{yydv}{dx} &+ y^0(P + aF + a(R + aH) + aa(V + aL)) \\ &+ y^1(Q + aG + v(R + aH) + a(T + aK) + 2av(V + aL)) \\ &+ y^2(S + aI + v(T + aK) + vv(V + aL)), \end{aligned}$$

nuncque efficiendum est, ut tota aequatio per yy dividi queat ideoque partes per y^0 et y^1 affectae evanescant. Ex parte ergo y^0 fieri oportet

$$P + aF + a(R + aH) + aa(V + aL) = 0,$$

ex parte autem y^1, quia v est nova variabilis in calculum inducta, hae duae conditiones nascuntur

$$Q + aG + a(T + aK) = 0 \quad \text{et} \quad R + aH + 2a(V + aL) = 0,$$

unde prima dabit

$$P + aF - aa(V + aL) = 0.$$

Conditiones ad istam reductionem requisitae sunt hae tres

$$\begin{aligned}
\text{I.}\quad & P + aF - aa(V + aL) = 0,\\
\text{II.}\quad & Q + aG + a(T + aK) = 0,\\
\text{III.}\quad & R + aH + 2a(V + aL) = 0,
\end{aligned}$$

unde vel P, Q et R vel F, G et H commode definiuntur.

His autem conditionibus stabilitis totum negotium ad resolutionem huius aequationis revocatur

$$0 = \frac{dv}{dx} + S + aI + v(T + aK) + vv(V + aL),$$

quae duas tantum continet variabiles x et v, ex qua v per x determinari oportet, cum deinde posito $z = a + yv$ prima aequatio induat hanc formam

$$0 = \frac{dy}{dx} + F + aH + aaL + y(G + Hv + aK + 2aLv) + yy(I + Kv + Lvv),$$

secunda vero istam

$$\begin{aligned}
0 = \frac{yydv}{dx} - \frac{ady}{dx} + P + aR + aaV + y(Q + Rv + aT + 2aVv)&\\
+ yy(S + Tv + Vvv),&
\end{aligned}$$

seu hinc superiorem per yy multiplicatam subtrahendo

$$\begin{aligned}
0 = \frac{-ady}{dx} + P + aR + aaV + y(Q + Rv + aT + 2aVv)&\\
- yy(Ia + aKv + aLvv),&
\end{aligned}$$

quae quidem cum illa congruit, ut natura rei postulat.

COROLLARIUM 1

376. Si ergo huiusmodi binae aequationes fuerint propositae

$$0 = \frac{dy}{dx} + F + Gy + Hz + Iyy + Kyz + Lzz,$$

$$\begin{aligned}
0 = \frac{ydz - zdy}{dx} - aF - aGy - aHz + Syy + Tyz + Vzz&\\
+ a^3L - aaKy - 2aaLz&\\
+ aaV - aTy - 2aVz,&
\end{aligned}$$

facto $z = a + yv$ primo resolvi debet haec aequatio

$$0 = \frac{dv}{dx} + S + aI + v(T + aK) + vv(V + aL),$$

unde definita v per x hanc aequationem tractari oportet

$$0 = \frac{dy}{dx} + F + aH + aaL + y(G + aK) + yy(I + Kv + Lvv) + vy(H + 2aL),$$

quo facto habebitur quoque $z = a + vy$.

COROLLARIUM 2

377. Si $F = A$, $K = 0$, $L = 0$, $H = -2b$, $V = b$ et $T = -G$, casus supra § 374 tractatus resultat harum aequationum

$$0 = \frac{dy}{dx} + A + Gy - 2bz + Iyy,$$

$$0 = \frac{ydz - zdy}{dx} - aA + aab + Syy - Gyz + bzz,$$

ubi G, I et S sunt functiones quaecunque ipsius x, et resolutio ita se habet, ut posito $z = a + yv$ hae aequationes successive debeant expediri

$$0 = \frac{dv}{dx} + S + aI - Gv + bvv$$

et

$$0 = \frac{dy}{dx} + A - 2ab + y(G - 2bv) + Iyy.$$

COROLLARIUM 3

378. Evidens est postremam aequationem nulla laborare difficultate etiam in genere, dum sit

$$F + aH + aaL = 0;$$

prioris autem solutio in promtu est, si sit vel $S + aI = 0$ vel $V + aL = 0$.

CALCVLI INTEGRALIS
LIBER POSTERIOR.

PARS PRIMA

SEV

INVESTIGATIO FVNCTIONVM DVARVM VARIABILIVM EX DATA DIFFERENTIALIVM CVIVSVIS GRADVS RELATIONE.

SECTIO TERTIA

INVESTIGATIO DVARVM VARIABILIVM FVNCTIONVM EX DATA DIFFERENTIALIVM TERTII ALTIORVMQVE GRADVVM RELATIONE.

CAPUT I

DE RESOLUTIONE AEQUATIONUM SIMPLICISSIMARUM UNICAM FORMULAM DIFFERENTIALEM INVOLVENTIUM

PROBLEMA 61

379. *Indolem functionis binarum variabilium x et y indagare, si eius quaepiam formula differentialis tertii gradus evanescat.*

SOLUTIO

Sit z functio illa quaesita, et cum eius sint quatuor formulae differentiales tertii gradus

$$\left(\frac{d^3 z}{dx^3}\right), \quad \left(\frac{d^3 z}{dx^2\, dy}\right), \quad \left(\frac{d^3 z}{dx\, dy^2}\right) \quad \text{et} \quad \left(\frac{d^3 z}{dy^3}\right),$$

prout quaelibet harum nihilo aequalis statuitur, totidem habemus casus evolvendos.

I. Sit igitur primo $\left(\frac{d^3 z}{dx^3}\right) = 0$ et sumta y constante prima integratio praebet

$$\left(\frac{ddz}{dx^2}\right) = \Gamma : y;$$

tum simili modo secunda integratio dat

$$\left(\frac{dz}{dx}\right) = x\Gamma : y + \Delta : y,$$

unde tandem fit

$$z = \frac{1}{2} xx\Gamma : y + x\Delta : y + \Sigma : y,$$

ubi $\Gamma : y$, $\Delta : y$ et $\Sigma : y$ denotant functiones quascunque ipsius y, ita ut ob triplicem integrationem tres functiones arbitrariae in calculum sint ingressae, ut rei natura postulat.

II. Sit $\left(\frac{d^3 z}{dx^2 dy}\right) = 0$ ac primo bis integrando per solius x variabilitatem reperitur ut ante

$$\left(\frac{dz}{dy}\right) = x\Gamma' : y + \Delta' : y;$$

nunc autem sola y pro variabili habita adipiscimur

$$z = x\Gamma : y + \Delta : y + \Sigma : x,$$

quandoquidem apices signis functionum inscripti hic semper hunc habent significatum, ut sit

$$\int dy \Gamma' : y = \Gamma : y \quad \text{et} \quad \int dy . \Delta' : y = \Delta : y.$$

III. Sit $\left(\frac{d^3 z}{dx dy^2}\right) = 0$, et quia hic casus a praecedente non differt, nisi quod binae variabiles x et y inter se sint permutatae, integrale quaesitum est

$$z = y\Gamma : x + \Delta : x + \Sigma : y.$$

IV. Sit $\left(\frac{d^3 z}{dy^3}\right) = 0$ et ob similem permutationem ex casu primo intelligitur fore

$$z = \frac{1}{2} yy\Gamma : x + y . \Delta : x + \Sigma : x.$$

COROLLARIUM 1

380. Tres functiones arbitrariae hic per triplicem integrationem ingressae sunt vel ipsius x vel ipsius y tantum; omnes tres sunt ipsius y tantum casu primo $\left(\frac{d^3 z}{dx^3}\right) = 0$, ipsius x vero tantum casu quarto $\left(\frac{d^3 z}{dy^3}\right) = 0$; duae vero sunt ipsius y et una ipsius x casu secundo $\left(\frac{d^3 z}{dx^2 dy}\right) = 0$; contra autem duae ipsius x et una ipsius y casu tertio $\left(\frac{d^3 z}{dx dy^2}\right) = 0$.

COROLLARIUM 2

381. Porro observasse iuvabit, si eiusdem variabilis, puta x, duae pluresve occurrant functiones arbitrariae, unam quidem absolute poni, alteram per y multiplicari, tertiam vero, si adsit, per $\frac{1}{2}yy$ seu, quod eodem redit, per yy multiplicatam accedere.

COROLLARIUM 3

382. Perpetuo autem tenendum est has functiones ita arbitrio nostro relinqui, ut etiam functiones discontinuae seu nulla continuitatis lege contentae non excludantur. Scilicet si libero manus tractu linea quaecunque describatur, applicata respondens abscissae x huiusmodi functionem $\Gamma : x$ referet.

SCHOLION 1

383. Minus hic immorandum arbitror transformationi formularum differentialium altioris gradus, dum loco binarum variabilium x et y aliae quaecunque in calculum introducuntur, quoniam in genere expressiones nimis fierent complicatae vixque ullum usum habiturae, tum vero imprimis, quod methodus has transformationes inveniendi iam supra (§ 229) satis luculenter est tradita. Casum tantum simpliciorem, quo binae novae variabiles t et u loco x et y introducendae ita accipiuntur, ut sit

$$t = \alpha x + \beta y \quad \text{et} \quad u = \gamma x + \delta y,$$

hic quoque ad formulas differentiales altiores accommodabo. Cum igitur viderimus esse [§ 233] pro formulis primi gradus

$$\left(\frac{dz}{dx}\right) = \alpha\left(\frac{dz}{dt}\right) + \gamma\left(\frac{dz}{du}\right), \quad \left(\frac{dz}{dy}\right) = \beta\left(\frac{dz}{dt}\right) + \delta\left(\frac{dz}{du}\right)$$

et pro formulis secundi gradus

$$\left(\frac{ddz}{dx^2}\right) = \alpha^2\left(\frac{ddz}{dt^2}\right) + 2\alpha\gamma\left(\frac{ddz}{dtdu}\right) + \gamma^2\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right) = \alpha\beta\left(\frac{ddz}{dt^2}\right) + (\alpha\delta + \beta\gamma)\left(\frac{ddz}{dtdu}\right) + \gamma\delta\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right) = \beta^2\left(\frac{ddz}{dt^2}\right) + 2\beta\delta\left(\frac{ddz}{dtdu}\right) + \delta^2\left(\frac{ddz}{du^2}\right),$$

erit pro formulis tertii gradus

$$\left(\frac{d^3z}{dx^3}\right) = \alpha^3\left(\frac{d^3z}{dt^3}\right) + 3\alpha^2\gamma\left(\frac{d^3z}{dt^2du}\right) + 3\alpha\gamma^2\left(\frac{d^3z}{dtdu^2}\right) + \gamma^3\left(\frac{d^3z}{du^3}\right),$$

$$\left(\frac{d^3z}{dx^2dy}\right) = \alpha^2\beta\left(\frac{d^3z}{dt^3}\right) + (\alpha^2\delta + 2\alpha\beta\gamma)\left(\frac{d^3z}{dt^2du}\right) + (\beta\gamma^2 + 2\alpha\gamma\delta)\left(\frac{d^3z}{dtdu^2}\right) + \gamma^2\delta\left(\frac{d^3z}{du^3}\right),$$

$$\left(\frac{d^3z}{dxdy^2}\right) = \alpha\beta^2\left(\frac{d^3z}{dt^3}\right) + (\beta^2\gamma + 2\alpha\beta\delta)\left(\frac{d^3z}{dt^2du}\right) + (\alpha\delta^2 + 2\beta\gamma\delta)\left(\frac{d^3z}{dtdu^2}\right) + \gamma\delta^2\left(\frac{d^3z}{du^3}\right),$$

$$\left(\frac{d^3z}{dy^3}\right) = \beta^3\left(\frac{d^3z}{dt^3}\right) + 3\beta^2\delta\left(\frac{d^3z}{dt^2du}\right) + 3\beta\delta^2\left(\frac{d^3z}{dtdu^2}\right) + \delta^3\left(\frac{d^3z}{du^3}\right)$$

et pro formulis quarti gradus

	$\left(\frac{d^4z}{dt^4}\right)$	$\left(\frac{d^4z}{dt^3du}\right)$	$\left(\frac{d^4z}{dt^2du^2}\right)$	$\left(\frac{d^4z}{dtdu^3}\right)$	$\left(\frac{d^4z}{du^4}\right)$
$\left(\frac{d^4z}{dx^4}\right) =$	α^4	$4\alpha^3\gamma$	$6\alpha^2\gamma^2$	$4\alpha\gamma^3$	γ^4
$\left(\frac{d^4z}{dx^3dy}\right) =$	$\alpha^3\beta$	$\alpha^3\delta + 3\alpha^2\beta\gamma$	$3\alpha^2\gamma\delta + 3\alpha\beta\gamma^2$	$3\alpha\gamma^2\delta + \beta\gamma^3$	$\gamma^3\delta$
$\left(\frac{d^4z}{dx^2dy^2}\right) =$	$\alpha^2\beta^2$	$2\alpha^2\beta\delta + 2\alpha\beta^2\gamma$	$\alpha^2\delta^2 + 4\alpha\beta\gamma\delta + \beta^2\gamma^2$	$2\alpha\gamma\delta^2 + 2\beta\gamma^2\delta$	$\gamma^2\delta^2$
$\left(\frac{d^4z}{dxdy^3}\right) =$	$\alpha\beta^3$	$3\alpha\beta^2\delta + \beta^3\gamma$	$3\alpha\beta\delta^2 + 3\beta^2\gamma\delta$	$\alpha\delta^3 + 3\beta\gamma\delta^2$	$\gamma\delta^3$
$\left(\frac{d^4z}{dy^4}\right) =$	β^4	$4\beta^3\delta$	$6\beta^2\delta^2$	$4\beta\delta^3$	δ^4

unde simul lex pro altioribus gradibus elucet; pro formula scilicet generali $\left(\frac{d^{m+n}z}{dx^m dy^n}\right)$ hi coefficientes iidem sunt, qui oriuntur ex evolutione huius formae $(\alpha+\gamma v)^m(\beta+\delta v)^n$, siquidem termini secundum potestates ipsius v disponantur.

SCHOLION 2

384. Haud alienum fore arbitror evolutionem istius formulae ex principiis ante[1]) stabilitis accuratius docere.

---

1) *Institutiones calculi differentialis*, partis posterioris § 201, *Leonhardi Euleri Opera omnia*, series I, vol. 10, p. 398. F. E.

Sit igitur

$$s = (\alpha + \gamma v)^m (\beta + \delta v)^n$$

ac ponatur

$$s = A + Bv + Cv^2 + Dv^3 + Ev^4 + Fv^5 + \text{etc.},$$

ubi quidem primo patet esse $A = \alpha^m \beta^n$; pro reliquis vero coefficientibus inveniendis sumtis differentialibus logarithmorum habebimus

$$\frac{ds}{s\,dv} = \frac{m\gamma}{\alpha + \gamma v} + \frac{n\delta}{\beta + \delta v}$$

ideoque

$$\frac{ds}{dv}(\alpha\beta + (\alpha\delta + \beta\gamma)v + \gamma\delta vv) - s(m\beta\gamma + n\alpha\delta + (m+n)\gamma\delta v) = 0;$$

ubi si loco s series assumta substituatur, orietur haec aequatio

$$\begin{array}{llllll}
0 = \alpha\beta B & + 2\alpha\beta Cv & + 3\alpha\beta Dv^2 & + 4\alpha\beta Ev^3 & + 5\alpha\beta Fv^4 + \text{etc.} \\
 & + \alpha\delta B & + 2\alpha\delta C & + 3\alpha\delta D & + 4\alpha\delta E \\
 & + \beta\gamma B & + 2\beta\gamma C & + 3\beta\gamma D & + 4\beta\gamma E \\
 & & + \gamma\delta B & + 2\gamma\delta C & + 3\gamma\delta D \\
- m\beta\gamma A & - m\beta\gamma B & - m\beta\gamma C & - m\beta\gamma D & - m\beta\gamma E \\
- n\alpha\delta A & - n\alpha\delta B & - n\alpha\delta C & - n\alpha\delta D & - n\alpha\delta E \\
 & -(m+n)\gamma\delta A & -(m+n)\gamma\delta B & -(m+n)\gamma\delta C & -(m+n)\gamma\delta D
\end{array}$$

unde quilibet coefficiens ex praecedentibus ita definitur

$$A = \alpha^m \beta^n,$$

$$B = \frac{m\beta\gamma + n\alpha\delta}{\alpha\beta} A,$$

$$C = \frac{(m-1)\beta\gamma + (n-1)\alpha\delta}{2\alpha\beta} B + \frac{(m+n)\gamma\delta}{2\alpha\beta} A,$$

$$D = \frac{(m-2)\beta\gamma + (n-2)\alpha\delta}{3\alpha\beta} C + \frac{(m+n-1)\gamma\delta}{3\alpha\beta} B,$$

$$E = \frac{(m-3)\beta\gamma + (n-3)\alpha\delta}{4\alpha\beta} D + \frac{(m+n-2)\gamma\delta}{4\alpha\beta} C$$

etc.

His igitur coefficientibus inventis si ponatur

$$t = \alpha x + \beta y \quad \text{et} \quad u = \gamma x + \delta y,$$

transformatio formulae differentialis cuiuscunque ita se habebit, ut sit

$$\left(\frac{d^{m+n} z}{dx^m dy^n}\right) = A\left(\frac{d^{m+n} z}{dt^{m+n}}\right) + B\left(\frac{d^{m+n} z}{dt^{m+n-1} du}\right) + C\left(\frac{d^{m+n} z}{dt^{m+n-2} du^2}\right) + \text{etc.}$$

PROBLEMA 62

385. *Indolem functionis binarum variabilium x et y investigare, si eius formula differentialis cuiuscunque gradus evanescat.*

SOLUTIO

Ex iis, quae de formulis differentialibus tertii gradus nihilo aequatis ostendimus in praecedente problemate, satis perspicuum est solutionem huius problematis pro formulis differentialibus quarti gradus ita se habere.

I. Si sit $\left(\frac{d^4 z}{dx^4}\right) = 0$, erit

$$z = x^3 \Gamma : y + x^2 \Delta : y + x \Sigma : y + \Theta : y.$$

II. Si sit $\left(\frac{d^4 z}{dx^3 dy}\right) = 0$, erit

$$z = x^2 \Gamma : y + x \Delta : y + \Sigma : y + \Theta : x.$$

III. Si sit $\left(\frac{d^4 z}{dx^2 dy^2}\right) = 0$, erit

$$z = x \Gamma : y + \Delta : y + y \Sigma : x + \Theta : x.$$

IV. Si sit $\left(\frac{d^4 z}{dx\, dy^3}\right) = 0$, erit

$$z = \Gamma : y + y^2 \Delta : x + y \Sigma : x + \Theta : x.$$

V. Si sit $\left(\frac{d^4 z}{dy^4}\right) = 0$, erit

$$z = y^3 \Gamma : x + y^2 \Delta : x + y \Sigma : x + \Theta : x;$$

unde simul progressus ad altiores gradus est manifestus.

COROLLARIUM 1

386. Cum hic quatuor functiones arbitrariae occurrant, totidem scilicet, quot integrationes institui oportet, in hoc ipso criterium integrationis completae continetur.

COROLLARIUM 2

387. Quin etiam vicissim facile ostenditur formas inventas aequationi propositae satisfacere. Sic cum pro casu tertio invenerimus

$$z = x\Gamma : y + \Delta : y + y\Sigma : x + \Theta : x,$$

differentiando hinc colligimus

$$\text{primo} \quad \left(\frac{dz}{dx}\right) = \Gamma : y + y\Sigma' : x + \Theta' : x,$$

$$\text{deinde} \quad \left(\frac{ddz}{dx^2}\right) = y\Sigma'' : x + \Theta'' : x,$$

$$\text{tertio} \quad \left(\frac{d^3z}{dx^2 dy}\right) = \Sigma'' : x$$

et

$$\text{quarto} \quad \left(\frac{d^4z}{dx^2 dy^2}\right) = 0$$

eodemque pervenitur, quocunque ordine differentiationes vel solam x vel solam y variabilem sumendo instituantur.

SCHOLION 1

388. Hactenus unam formulam differentialem nihilo esse aequalem assumsimus; calculus autem perinde succedit, si huiusmodi formula functioni cuicunque ipsarum x et y aequalis statuatur, quemadmodum in sequentibus problematibus sum ostensurus. Hoc tantum inculcandum censeo, si V fuerit functio quaecunque binarum variabilium x et y, tum $\int V dx$ id denotare integrale, quod obtinetur, si sola x pro variabili habeatur, in hac vero formula $\int V dy$ solam y pro variabili haberi; quod idem tenendum est de integrationibus repetitis veluti $\int dx \int V dx$, ubi in utraque sola x variabilis assumitur, in hac vero $\int dy \int V dx$, postquam integrale $\int V dx$ ex sola ipsius x variabi-

litate fuerit erutum, tum in altera integratione $\int dy \int V dx$ solam y variabilem accipiendam esse. Et cum perinde, utra integratio prior instituatur, etiam hoc discrimen e modo signandi tolli potest hocque integrale geminatum ita $\int\int V dx dy$ exhiberi; hincque intelligitur, quomodo has formulas

$$\int\int\int V dx^2 dy \quad \text{seu} \quad \int^3 V dx^2 dy \quad \text{et} \quad \int^{m+n} V dx^m dy^n$$

interpretari oporteat; hic scilicet signo integrationis $\int$ indices suffigimus, prorsus uti signo differentiationis d suffigi solent, quippe qui indicant, quoties integratio sit repetenda.

SCHOLION 2

389. Singulas has integrationes repetendas ita institui hic assumimus, ut nulla relatio inter binas variabiles x et y in subsidium vocetur; quae circumstantia eo diligentius est animadvertenda, cum vulgo, ubi talibus integrationibus opus est, calculus prorsus diverso modo institui debeat. Quodsi enim proposito quopiam corpore geometrico eius soliditas seu superficies sit investiganda, per duplicem integrationem huiusmodi formula $\int\int V dx dy$ evolvi debet existente V certa functione ipsarum x et y; ubi quidem primo quaeritur integrale $\int V dy$ spectata x ut constante, at absoluta integratione ad terminos integrationi praescriptos respici oportet, dum scilicet altero praescribitur, ut hoc integrale $\int V dy$ evanescat posito $y = 0$, altero vero id eo usque extendendum est, donec y datae cuipiam functioni ipsius x aequetur. Tum vero, postquam hoc integrale $\int V dy$ isto modo fuerit determinatum, altera demum integratio formulae $dx \int V dy$ suscipitur, in qua quantitas y non amplius inest, dum eius loco certa quaepiam functio ipsius x est substituta eaque formula iam revera unicam variabilem x complectitur. Hic ergo prima integratione absoluta variabilis y in functionem ipsius x abire est censenda, quam propterea in altera integratione, ubi x est variabilis, minime ut constantem spectare licebit. Ex quo patet hunc casum toto coelo esse diversum ab iis integrationibus repetendis, quas hic contemplamur; ad quem propterea hic eo minus respicimus, cum ista peculiaris ratio tantum in formula $\int\int V dx dy$ locum habere possit, reliquis vero, ubi alterum differentiale dx vel dy saepius repetitur, adeo adversetur. Quam ob causam hinc omnem relationem, quae forte peracta una integratione inter binas variabiles x et y statui posset, merito removemus.

PROBLEMA 63

390. *Si formula quaepiam differentialis tertii altiorisve gradus aequetur functioni cuicunque binarum variabilium x et y, indolem functionis z definire.*

SOLUTIO

Sit V functio quaecunque binarum variabilium x et y et incipientes a formulis tertii ordinis sit primo $\left(\frac{d^3z}{dx^3}\right) = V$ et posita sola x variabili erit

$$\left(\frac{ddz}{dx^2}\right) = \int V dx + \Gamma : y;$$

tum vero porro

$$\left(\frac{dz}{dx}\right) = \int dx \int V dx + x\Gamma : y + \varDelta : y = \int\int V dx^2 + x\Gamma : y + \varDelta : y$$

ac denique

$$z = \int^3 V dx^3 + \frac{1}{2}x^2\Gamma : y + x\varDelta : y + \Sigma : y.$$

Simili modo patet, si fuerit $\left(\frac{d^3z}{dx^2dy}\right) = V$, fore

$$z = \int^3 V dx^2 dy + x\Gamma : y + \varDelta : y + \Sigma : x,$$

ac si sit $\left(\frac{d^3z}{dxdy^2}\right) = V$, erit

$$z = \int^3 V dx dy^2 + \Gamma : y + y\varDelta : x + \Sigma : x;$$

si sit $\left(\frac{d^3z}{dy^3}\right) = V$, erit

$$z = \int^3 V dy^3 + y^2\Gamma : x + y\varDelta : x + \Sigma : x.$$

Eodem modo ad formulas altiorum graduum progredientes reperiemus, ut sequitur; si sit $\left(\frac{d^4z}{dx^4}\right) = V$, fore

$$z = \int^4 V dx^4 + x^3\Gamma : y + x^2\varDelta : y + x\Sigma : y + \Theta : y;$$

si sit $\left(\frac{d^4z}{dx^3dy}\right) = V$, fore

$$z = \int^4 V dx^3 dy + x^2\Gamma : y + x\varDelta : y + \Sigma : y + \Theta : x;$$

si sit $\left(\frac{d^4z}{dx^2dy^2}\right) = V$, fore

$$z = \int^4 Vdx^2dy^2 + x\Gamma : y + \varDelta : y + y\Sigma : x + \Theta : x;$$

si sit $\left(\frac{d^4z}{dxdy^3}\right) = V$, fore

$$z = \int^4 Vdxdy^3 + \Gamma : y + y^2\varDelta : x + y\Sigma : x + \Theta : x;$$

si sit $\left(\frac{d^4z}{dy^4}\right) = V$, fore

$$z = \int^4 Vdy^4 + y^3\Gamma : x + y^2\varDelta : x + y\Sigma : x + \Theta : x;$$

neque pro altioribus gradibus res eget ulteriori explicatione.

COROLLARIUM 1

391. Quemadmodum signum integrationis in primo libro usitatum iam per se involvit constantem per integrationem ingredientem, ita quoque hic functiones arbitrariae per integrationem ingressae iam in formula integrali involvi sunt censendae, ita ut non sit opus eas exprimere.

COROLLARIUM 2

392. Sufficit ergo pro aequatione $\left(\frac{d^3z}{dx^3}\right) = V$ integrale triplicatum hoc modo dedisse $z = \int^3 Vdx^3$, quae forma iam potestate complectitur partes supra adiectas

$$xx\Gamma : y + x\varDelta : y + \Sigma : y;$$

quod idem de reliquis est tenendum.

COROLLARIUM 3

393. Si ergo in genere haec habeatur aequatio

$$\left(\frac{d^{m+n}z}{dx^m dy^n}\right) = V,$$

eius integrale statim hoc modo exhibetur

$$z = \int^{m+n} Vdx^m dy^n,$$

quae potestate iam involvit omnes illas functiones arbitrarias numero $m + n$ per totidem integrationes invectas.

SCHOLION

394. Hi casus utique sunt simplicissimi, qui ad hoc caput referendi videntur, pro magis autem complicatis vix certa praecepta tradere licet, cum ista calculi integralis pars vix adhuc coli sit coepta. Interim tamen iam intelligitur, si aequationes magis complicatas ope cuiusdam transformationis ad has simplicissimas revocare liceat, etiam earum integrationem in promtu esse futuram; quod quidem negotium hic non copiosius persequendum videtur. Progredior igitur ad casus magis reconditos eosque ita comparatos, ut ope aequationum inferiorum ordinum expediri queant, unde quidem insignis methodus satis late patens colligi poterit, qua saepius haud sine successu uti licebit. Neque tamen in hac pertractatione nimis diffusum esse convenit, sed sufficiet praecipuos fontes adhuc quidem cognitos patefecisse.

CAPUT II

DE INTEGRATIONE AEQUATIONUM ALTIORUM PER REDUCTIONEM AD INFERIORES

PROBLEMA 64

395. *Proposita hac aequatione tertii gradus* $\left(\frac{d^3z}{dx^3}\right) = a^3z$ *indolem functionis* z *investigare.*

SOLUTIO

Fingatur huic aequationi satisfacere haec simplicior primi gradus

$$\left(\frac{dz}{dx}\right) = nz,$$

et cum hinc differentiando obtineatur

$$\left(\frac{ddz}{dx^2}\right) = n\left(\frac{dz}{dx}\right) = nnz$$

hincque porro

$$\left(\frac{d^3z}{dx^3}\right) = nn\left(\frac{dz}{dx}\right) = n^3z,$$

evidens est quaesito satisfieri, dum sit $n^3 = a^3$, id quod triplici modo evenire potest:

$$\text{I. } n = a, \quad \text{II. } n = \frac{-1+\sqrt{-3}}{2}a, \quad \text{III. } n = \frac{-1-\sqrt{-3}}{2}a.$$

Pro quolibet ergo valore quaeratur integrale completum aequationis $\left(\frac{dz}{dx}\right) = nz$ et tria haec integralia coniuncta praebebunt integrale completum aequationis

propositae. Cum autem in aequatione $\left(\frac{dz}{dx}\right) = nz$ quantitas y constans sumatur, erit

$$dz = nzdx \quad \text{seu} \quad \frac{dz}{z} = ndx,$$

unde fit

$$lz = nx + l\Gamma : y \quad \text{seu} \quad z = e^{nx}\Gamma : y.$$

Tribuantur iam ipsi n terni valores eritque pro aequatione proposita

$$z = e^{ax}\Gamma : y + e^{\frac{-1+\sqrt{-3}}{2}ax}\varDelta : y + e^{\frac{-1-\sqrt{-3}}{2}ax}\Sigma : y.$$

Cum autem sit

$$e^{m\sqrt{-1}} = \cos. m + \sqrt{-1}\cdot\sin. m,$$

erit functionum arbitrariarum formam mutando

$$z = e^{ax}\Gamma : y + e^{-\frac{1}{2}ax}\cos.\frac{ax\sqrt{3}}{2}\cdot\varDelta : y + e^{-\frac{1}{2}ax}\sin.\frac{ax\sqrt{3}}{2}\cdot\Sigma : y.$$

COROLLARIUM 1

396. Integrale hoc etiam ita repraesentari potest

$$z = e^{ax}\Gamma : y + e^{-\frac{1}{2}ax}\varDelta : y\cdot\cos.\left(\frac{ax\sqrt{3}}{2} + Y\right)$$

denotante Y functionem quamcunque ipsius y.

COROLLARIUM 2

397. Quia tribus integrationibus est opus et in singulis quantitas y ut constans tractatur, secundum praecepta libri primi haec aequatio $d^3z = a^3zdx^3$ resolvatur[1]) et loco trium constantium functiones quaecunque ipsius y introducantur, unde eadem solutio elicitur.

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1) *Institutionum calculi integralis* vol. II, § 1136, LEONHARDI EULERI *Opera omnia*, series I, vol. 12, p. 316. F. E.

PROBLEMA 65

398. *Proposita hac aequatione cuiuscunque gradus*

$$Pz + Q\left(\frac{dz}{dx}\right) + R\left(\frac{ddz}{dx^2}\right) + S\left(\frac{d^3z}{dx^3}\right) + T\left(\frac{d^4z}{dx^4}\right) + \text{etc.} = 0,$$

ubi litterae P, Q, R, S, T etc. functiones denotant quascunque binarum variabilium x et y, indolem functionis z definire.

SOLUTIO

Cum in omnibus integrationibus instituendis quantitas y perpetuo ut constans spectetur, haec aequatio inter duas tantum variabiles x et z consistere est censenda. Quare per praecepta libri primi[1]) haec tractanda erit aequatio

$$Pz + \frac{Qdz}{dx} + \frac{Rddz}{dx^2} + \frac{Sd^3z}{dx^3} + \frac{Td^4z}{dx^4} + \text{etc.} = 0;$$

cuius resolutio si succedat, tantum opus est, ut loco constantium per singulas integrationes invectarum functiones quaecunque ipsius y scribantur; sicque habebitur integrale desideratum idque completum, siquidem hanc aequationem complete integrare licuerit.

COROLLARIUM 1

399. Si ergo litterae P, Q, R, S etc. sint constantes vel solam variabilem y involvant, integratio semper succedit, quoniam in primo libro huiusmodi aequationes in genere integrare docuimus.

COROLLARIUM 2

400. Deinde etiam resolutio succedit huius aequationis[2])

$$Az + Bx\left(\frac{dz}{dx}\right) + Cx^2\left(\frac{ddz}{dx^2}\right) + Dx^3\left(\frac{d^3z}{dx^3}\right) + \text{etc.} = 0,$$

sive litterae A, B, C etc. sint constantes sive functiones ipsius y tantum.

1) *Institutionum calculi integralis* liber I, pars II, sectio I, cap. III—XI, et sectio II, cap. II, LEONHARDI EULERI *Opera omnia* series I, vol. 12, p. 47—270 et 296. F. E.

2) Vide ibidem sectionis II cap. V, p. 381. F. E.

COROLLARIUM 3

401. Tum vero etiam, si hae formae non sint aequales nihilo, sed functioni cuicunque ipsarum x et y aequentur, resolutio nihilo minus succedit per ea, quae in postremis capitibus libri primi sunt exposita.

SCHOLION

402. Haec etiam multo latius extendi possunt ad omnes plane aequationes, in quibus nullae aliae formulae differentiales praeter has

$$\left(\frac{dz}{dx}\right), \quad \left(\frac{ddz}{dx^2}\right), \quad \left(\frac{d^3z}{dx^3}\right) \quad \text{etc.},$$

quae solam x ut variabilem implicant, occurrunt. Quomodocunque enim istae formulae cum quantitatibus finitis x, y et z fuerint complicatae, aequatio semper ad librum primum pertinere est censenda, quoniam in omnibus integrationibus instituendis quantitas y perpetuo ut constans tractatur. Confectis demum integrationibus discrimen in hoc consistit, ut loco constantium arbitrariarum functiones arbitrariae ipsius y in calculum introducantur. Superfluum foret hic monere, quae de altera variabilium y sunt dicta, etiam de altera x esse intelligenda.

PROBLEMA 66

403. *Proposita hac aequatione*

$$\left(\frac{ddz}{dx^2}\right) + b\left(\frac{ddz}{dxdy}\right) - 2a\left(\frac{dz}{dx}\right) - ab\left(\frac{dz}{dy}\right) + aaz = 0$$

investigare indolem functionis z.

SOLUTIO

Facile patet huic aequationi satisfacere hanc aequationem simplicem $\left(\frac{dz}{dx}\right) = az$, unde fit $z = e^{ax}$. Statuamus ergo $z = e^{ax}v$ eritque

$$\left(\frac{dz}{dx}\right) = e^{ax}\left(av + \left(\frac{dv}{dx}\right)\right), \quad \left(\frac{dz}{dy}\right) = e^{ax}\left(\frac{dv}{dy}\right)$$

hincque

$$\left(\frac{ddz}{dx^2}\right) = e^{ax}\left(aav + 2a\left(\frac{dv}{dx}\right) + \left(\frac{ddv}{dx^2}\right)\right)$$

et

$$\left(\frac{ddz}{dxdy}\right) = e^{ax}\left(a\left(\frac{dv}{dy}\right) + \left(\frac{ddv}{dx\,dy}\right)\right),$$

quibus valoribus substitutis et divisa aequatione per e^{ax} habebimus

$$\left(\frac{ddv}{dx^2}\right) + b\left(\frac{ddv}{dxdy}\right) = 0.$$

Quia nunc hic ubique occurrit $\left(\frac{dv}{dx}\right)$, faciamus $\left(\frac{dv}{dx}\right) = u$; erit

$$\left(\frac{du}{dx}\right) + b\left(\frac{du}{dy}\right) = 0,$$

cuius integrale est $f:(y - bx) = u$ [§ 79]; scribamus ergo

$$u = \left(\frac{dv}{dx}\right) = -bΓ':(y - bx),$$

ut prodeat $v = Γ:(y - bx) + Δ:y$, ideoque integrale quaesitum erit

$$z = e^{ax}(Γ:(y - bx) + Δ:y),$$

quae forma ob duas functiones arbitrarias utique est integrale completum.

PROBLEMA 67

404. *Proposita hac aequatione*

$$0 = (a + 2b)z - (2a + 3b)\left(\frac{dz}{dx}\right) + c\left(\frac{dz}{dy}\right) + a\left(\frac{ddz}{dx^2}\right) - 2c\left(\frac{ddz}{dxdy}\right)$$
$$+ b\left(\frac{d^3z}{dx^3}\right) + c\left(\frac{d^3z}{dx^2dy}\right)$$

indolem functionis z investigare.

SOLUTIO

Aequatio haec ita est comparata, ut ei manifesto satisfaciat $z = e^x$; statuamus ergo $z = e^x v$ eritque

$$\left(\frac{dz}{dx}\right) = e^x\left(v + \left(\frac{dv}{dx}\right)\right), \quad \left(\frac{dz}{dy}\right) = e^x\left(\frac{dv}{dy}\right),$$

$$\left(\frac{ddz}{dx^2}\right) = e^x\left(v + 2\left(\frac{dv}{dx}\right) + \left(\frac{ddv}{dx^2}\right)\right), \quad \left(\frac{ddz}{dxdy}\right) = e^x\left(\left(\frac{dv}{dy}\right) + \left(\frac{ddv}{dxdy}\right)\right),$$

$$\left(\frac{d^3z}{dx^3}\right) = e^x\left(v + 3\left(\frac{dv}{dx}\right) + 3\left(\frac{ddv}{dx^2}\right) + \left(\frac{d^3v}{dx^3}\right)\right),$$

$$\left(\frac{d^3z}{dx^2dy}\right) = e^x\left(\left(\frac{dv}{dy}\right) + 2\left(\frac{ddv}{dxdy}\right) + \left(\frac{d^3v}{dx^2dy}\right)\right),$$

quibus valoribus substitutis emergit haec satis simplex aequatio

$$0 = (a + 3b)\left(\frac{ddv}{dx^2}\right) + b\left(\frac{d^3v}{dx^3}\right) + c\left(\frac{d^3v}{dx^2dy}\right),$$

in qua commode evenit, ut in singulis terminis formula $\left(\frac{ddv}{dx^2}\right)$ contineatur quare posito $\left(\frac{ddv}{dx^2}\right) = u$ prodit haec aequatio primi gradus

$$0 = (a + 3b)u + b\left(\frac{du}{dx}\right) + c\left(\frac{du}{dy}\right),$$

ex qua patet, si ponatur $du = pdx + qdy$, esse debere

$$(a + 3b)u + bp + cq = 0,$$

quae ita resolvitur.

Cum posito $a + 3b = f$ sit

$$q = -\frac{bp}{c} - \frac{fu}{c},$$

erit

$$du = pdx - \frac{bpdy}{c} - \frac{fudy}{c}$$

seu

$$dx - \frac{bdy}{c} = \frac{1}{p}\left(du + \frac{fudy}{c}\right) = \frac{u}{p}\left(\frac{du}{u} + \frac{fdy}{c}\right),$$

sicque necesse est, ut sit $\frac{u}{p}$ functio ipsius $x - \frac{by}{c}$, unde fit

$$lu + \frac{fy}{c} = f:(cx - by)$$

et

$$u = e^{\frac{-fy}{c}}\Gamma'':\left(x - \frac{by}{c}\right) = \left(\frac{ddv}{dx^2}\right).$$

Iam ob y constans spectandum prima integratio dat

$$\left(\frac{dv}{dx}\right)=e^{\frac{-fy}{c}}\Gamma':\left(x-\frac{by}{c}\right)+\Delta:y$$

et altera

$$v=e^{\frac{-fy}{c}}\Gamma:\left(x-\frac{by}{c}\right)+x\Delta:y+\Sigma:y.$$

Quare posito $a+3b=f$ aequationis propositae integrale completum est

$$z=e^{x-\frac{fy}{c}}\Gamma:\left(x-\frac{by}{c}\right)+e^{x}x\Delta:y+e^{x}\Sigma:y.$$

PROBLEMA 67a[1])

405. *Proposita hac aequatione differentiali tertii gradus*

$$0=Pz-3P\left(\frac{dz}{dx}\right)+3P\left(\frac{ddz}{dx^2}\right)-P\left(\frac{d^3z}{dx^3}\right)+Q\left(\frac{dz}{dy}\right)-2Q\left(\frac{ddz}{dxdy}\right)+Q\left(\frac{d^3z}{dx^2dy}\right),$$

ubi P et Q sint functiones quaecunque ipsarum x et y, investigare indolem functionis z.

SOLUTIO

Facta substitutione $z=e^x v$, quandoquidem ex data forma facile perspicitur valorem e^x loco z positum satisfacere, pervenitur ad hanc aequationem

$$-P\left(\frac{d^3v}{dx^3}\right)+Q\left(\frac{d^3v}{dx^2dy}\right)=0,$$

quae porro posito $\left(\frac{ddv}{dx^2}\right)=u$, ut sit $v=\iint u dx^2$, abit in hanc

$$-P\left(\frac{du}{dx}\right)+Q\left(\frac{du}{dy}\right)=0.$$

Statuamus $du=pdx+qdy$; erit $Qq=Pp$, hinc $q=\frac{Pp}{Q}$ ideoque

$$du=p\left(dx+\frac{P}{Q}dy\right),$$

1) Editio princeps: *Problema 68*; vide notam p. 302. F. E.

ex quo intelligitur quantitatem p ita comparatam esse debere, ut formula

$$dx + \frac{P}{Q}dy$$

per eam multiplicata integrabilis evadat. Quaeratur ergo multiplicator M formulam $Qdx + Pdy$ integrabilem reddens, ita ut sit

$$\int M(Qdx + Pdy) = s;$$

quam ergo functionem s ipsarum x et y inveniri posse assumo et ob

$$Qdx + Pdy = \frac{ds}{M}$$

habebimus $du = \frac{pds}{MQ}$, unde patet $\frac{p}{MQ}$ functionem denotare quantitatis s. Posito ergo $\frac{p}{MQ} = \Gamma' : s$ statim erit $u = \Gamma : s$ hincque $v = \int dx \int dx \Gamma : s$, in qua utraque integratione quantitas y ut constans spectatur. Quocirca resolutio problematis ita se habebit:

Pro formula differentiali $Qdx + Pdy$ quaeratur multiplicator M eam reddens integrabilem, ut sit

$$M(Qdx + Pdy) = ds,$$

et inventa hac ipsarum x et y functione s erit

$$z = e^x \int dx \int dx \Gamma : s + e^x x \Delta : y + e^x \Sigma : y.$$

SCHOLION

406. In istis aequationibus hoc commodi usu venit, ut facta substitutione $z = e^x v$ eiusmodi induant formam, quae facile porro ad speciem simplicem in prima sectione consideratam revocari queat; etiamsi enim differentialia tertii gradus non sint destructa, tamen reliqua membra ista e calculo excesserunt, ut deinceps nova substitutione $\left(\frac{ddv}{dx^2}\right) = u$ uti eiusque ope ad aequationem differentialem primi gradus pervenire licuerit. Unica igitur substitutio hoc praestitura fuisset, si statim posuissemus $z = e^x \int\int u dx^2$. Utinam praecepta haberentur, quorum ope huiusmodi substitutiones facile dignosci possent!

Interim postremo problemate multo latius patente in subsidium vocata § 209 resolvi poterit:

PROBLEMA 67 b[1])

407. *Proposita hac aequatione differentiali tertii gradus*

$$0 = (P + Q)z - (2P + 3Q)\left(\frac{dz}{dx}\right) + (P + 3Q)\left(\frac{ddz}{dx^2}\right) - Q\left(\frac{d^3z}{dx^3}\right)$$
$$- \qquad R\left(\frac{dz}{dy}\right) + \qquad 2R\left(\frac{ddz}{dxdy}\right) - R\left(\frac{d^3z}{dx^2dy}\right),$$

ubi P, Q et R sint functiones quaecunque datae ipsarum x et y, investigare indolem functionis z.

SOLUTIO

Eadem adhibita substitutione $z = e^x v$, qua hactenus sumus usi, aequatio proposita transmutatur in sequentem

$$0 = P\left(\frac{ddv}{dx^2}\right) - Q\left(\frac{d^3v}{dx^3}\right) - R\left(\frac{d^3v}{dx^2dy}\right),$$

ubi commode evenit, ut posito $\left(\frac{ddv}{dx^2}\right) = u$ ista resultet aequatio differentialis primi gradus

$$0 = Pu - Q\left(\frac{du}{dx}\right) - R\left(\frac{du}{dy}\right),$$

unde, qualis ipsarum x et y functio sit u, est inquirendum.

Ponamus esse $du = pdx + qdy$, et quia iam illa conditio praebet

$$Pu = Qp + Rq,$$

secundum artificium supra § 209 usurpatum formemus hinc tres sequentes aequationes

$$Ldu = Lpdx + Lqdy,$$
$$MPudx = MQpdx + MRqdx,$$
$$NPudy = NQpdy + NRqdy,$$

quae in unam summam collectae dabunt

$$Ldu + Pu(Mdx + Ndy) = p((L + MQ)dx + NQdy)$$
$$+ q((L + NR)dy + MRdx);$$

1) Editio princeps: *Problema 67*; in editione principe falso numeri 67 et 68 iterantur. F. E.

ubi cum tres quantitates L, M et N ab arbitrio nostro pendeant, inter eas statuatur primo eiusmodi relatio, ut binae partes posterioris membri communem obtineant factorem, sit scilicet

$$L + MQ : NQ = MR : L + NR \quad \text{seu} \quad L = -MQ - NR,$$

et habebimus

$$-du(MQ + NR) + Pu(Mdx + Ndy) = (Mq - Np)(Rdx - Qdy).$$

Quaeratur multiplicator T formulam $Rdx - Qdy$ reddens integrabilem, ut sit $T(Rdx - Qdy) = ds$, ex quo tam functio T quam s ut cognita spectari poterit, et quia nunc habemus

$$-du(MQ + NR) + Pu(Mdx + Ndy) = (Mq - Np)\frac{ds}{T}$$

seu

$$\frac{du}{u} - \frac{P(Mdx + Ndx)}{MQ + NR} = \frac{Np - Mq}{u(MQ + NR)} \cdot \frac{ds}{T},$$

nunc, cum P, Q, R sint functiones datae ipsarum x et y, probe notandum est inter binas nondum definitas M et N semper eiusmodi relationem statui posse, ut formula $\frac{P(Mdx + Ndy)}{MQ + NR}$ integrationem admittat; sit ergo eius integrale $= lw$, ita ut sit

$$Mdx + Ndy = \frac{MQ + NR}{P} \cdot \frac{dw}{w} \quad \text{et} \quad \frac{du}{u} = \frac{dw}{w} + \frac{Np - Mq}{Tu(MQ + NR)} ds.$$

Necesse ergo est quantitates p et q ita sint comparatae, ut fiat

$$\frac{Np - Mq}{Tu(MQ + NR)} = f' : s$$

hincque $lu = lw + f : s$. Loco $f : s$ scribamus $l\Gamma : s$, ut prodeat $u = w\Gamma : s$ ac propterea

$$v = \int dx \int w dx \Gamma : s + x\varDelta : y + \Sigma : y.$$

Consequenter

$$z = e^x \int dx \int w dx \Gamma : s + e^x x \varDelta : y + e^x \Sigma : y.$$

COROLLARIUM 1

408. Ad hanc ergo solutionem ex forma proposita statim eruendam primo quaeratur eiusmodi functio ipsarum x et y, quae vocetur s, ut sit

$$ds = T(Rdx - Qdy),$$

id quod expedietur multiplicatorem T investigando, quo formula differentialis $Rdx - Qdy$ integrabilis reddatur.

COROLLARIUM 2

409. Praeterea vero quoque quantitatem w investigari oportet. In hunc finem inter quantitates M et N eiusmodi rationem indagari convenit, ut fiat

$$\int \frac{P(Mdx + Ndy)}{MQ + NR} = lw,$$

quae quidem investigatio semper est concedenda.

SCHOLION

410. Cum statim totum negotium eo sit perductum, ut functio u ex hac aequatione definiri debeat

$$Pu = Q\left(\frac{du}{dx}\right) + R\left(\frac{du}{dy}\right),$$

sine ambagibus, quibus in solutione sum usus, solutio sequenti modo multo facilius absolvi poterit, id quod insigne supplementum in sectionem primam [§ 209] suppeditat.

Statuatur

$$\left(\frac{du}{dx}\right) = LMu \quad \text{et} \quad \left(\frac{du}{dy}\right) = LNu;$$

erit primo $P = L(MQ + NR)$, hinc $L = \frac{P}{MQ + NR}$, deinde ob

$$du = dx\left(\frac{du}{dx}\right) + dy\left(\frac{du}{dy}\right)$$

habebimus

$$\frac{du}{u} = L(Mdx + Ndy) = \frac{P(Mdx + Ndy)}{MQ + NR},$$

ubi M et N ita accipi oportet, ut integratio succedat, quod cum innumeris modis fieri possit, solutio hinc completa obtineri est aestimanda.

Verum dum casus integrationis particularis constet, multo commodius inde solutio completa sequenti ratione elicietur. Posito scilicet

$$\frac{dw}{w} = \frac{P(Mdx + Ndy)}{MQ + NR},$$

ita ut valor ipsius w pro u sumtus iam particulariter satisfaciat sitque

$$Pw = Q\left(\frac{dw}{dx}\right) + R\left(\frac{dw}{dy}\right),$$

statuamus pro valore completo $u = w\Gamma : s$ et facta substitutione consequimur

$$Pw\Gamma : s = Q\left(\frac{dw}{dx}\right)\Gamma : s + R\left(\frac{dw}{dy}\right)\Gamma : s + Qw\left(\frac{ds}{dx}\right)\Gamma' : s + Rw\left(\frac{ds}{dy}\right)\Gamma' : s,$$

quae aequatio subito in hanc contrahitur

$$Q\left(\frac{ds}{dx}\right) + R\left(\frac{ds}{dy}\right) = 0,$$

ex qua concludimus

$$\left(\frac{ds}{dx}\right) = TR \quad \text{et} \quad \left(\frac{ds}{dy}\right) = -TQ$$

ac propterea

$$ds = T(Rdx - Qdy),$$

unde patet hanc quantitatem s inveniri ex formula $Rdx - Qdy$, pro qua primo factor T eam reddens integrabilem quaeri, tum vero eius integrale pro s sumi debet. Imprimis igitur hic attendatur, quam concinne eandem solutionem elicere liceat, ad quam per tantas ambages perveneramus.

PROBLEMA 68

411. *Proposita hac aequatione differentiali quarti gradus*

$$\left(\frac{d^4 z}{dy^4}\right) = aa\left(\frac{ddz}{dx^2}\right)$$

functionis z inventionem saltem ad resolutionem aequationis simplicioris reducere.

SOLUTIO

Hanc aequationem attentius contemplanti mox patebit ei satisfacere huiusmodi simpliciorem

$$\left(\frac{ddz}{dy^2}\right) = b\left(\frac{dz}{dx}\right);$$

hinc enim per y differentiando fit

$$\left(\frac{d^3z}{dy^3}\right) = b\left(\frac{ddz}{dx\,dy}\right)$$

ac denuo eodem modo

$$\left(\frac{d^4z}{dy^4}\right) = b\left(\frac{d^3z}{dx\,dy^2}\right),$$

at ex ipsa assumta per x differentiata prodit

$$\left(\frac{d^3z}{dx\,dy^2}\right) = b\left(\frac{ddz}{dx^2}\right),$$

quo valore ibi inducto colligitur

$$\left(\frac{d^4z}{dy^4}\right) = bb\left(\frac{ddz}{dx^2}\right),$$

quae forma cum proposita congruit, dum sit $bb = aa$; quod cum duplici modo evenire queat, $b = +a$ et $b = -a$, postquam has aequationes simpliciores resolverimus

$$\left(\frac{ddz}{dy^2}\right) - a\left(\frac{dz}{dx}\right) = 0,$$

quae praebeat $z = P$,

$$\left(\frac{ddz}{dy^2}\right) + a\left(\frac{dz}{dx}\right) = 0,$$

quae praebeat $z = Q$, erit pro aequatione proposita $z = P + Q$, et quia tam P quam Q binas functiones arbitrarias involvit, integrale hoc modo inventum quatuor eiusmodi functiones complectetur ideoque erit completum.

COROLLARIUM 1

412. Solutiones particulares infinitae facile eliciuntur ponendo $z = e^{\mu x + \nu y}$. Facta enim substitutione fieri necesse est

$$\nu^4 = \mu\mu aa \quad \text{et} \quad \mu = \pm\frac{\nu\nu}{a}.$$

Sit $\nu = \lambda a$; erit $\mu = \pm\lambda\lambda a$ et integrale satisfaciens

$$z = e^{\lambda a(y \pm \lambda x)}.$$

COROLLARIUM 2

413. Poni etiam potest $z = e^{\mu x}\cos.(\nu y + \alpha)$, unde fit $\nu^4 = \mu\mu aa$ ut ante, ita ut alia forma integralium particularium sit

$$z = e^{\pm\lambda\lambda ax}\cos.(\lambda ay + \alpha).$$

Huiusmodi formulae infinitae coniunctae integrale completum quasi exhaurire sunt putandae.

COROLLARIUM 3

414. Eaedem solutiones reperiuntur ponendo generalius $z = XY$, unde fit

$$\frac{Xd^4Y}{dy^4} = \frac{aaYddX}{dx^2},$$

qua aequatione ita repraesentata

$$\frac{d^4Y}{Ydy^4} = \frac{aaddX}{Xdx^2}$$

utrumque membrum eidem constanti aequari debet.

SCHOLION

415. Aequatio autem, ad quam totum negotium reduximus,

$$\left(\frac{ddz}{dy^2}\right) = b\left(\frac{dz}{dx}\right)$$

ex earum est numero, quae nullo modo in genere resolvi posse videntur, ita ut in solutionibus particularibus acquiescere debeamus.

Aequatio autem proposita non in mera speculatione est posita, sed, quando laminarum elasticarum vibrationes quam minimae in genere investigantur, ad huiusmodi aequationem quarti gradus resolvendam pervenitur[1]),

1) Vide L. EULERI Commentationem 443 (indicis ENESTROEMIANI): *De motu vibratorio laminarum elasticarum, ubi plures novae vibrationum species hactenus non pertractatae evolvuntur*, Novi comment. acad. sc. Petrop. 17 (1772), 1773, § IX, p. 456; *LEONHARDI EULERI Opera omnia*, series II, vol. 9. F. E.

quae etiam causa est, quod haec quaestio non perinde atque cordarum vibrantium in genere adhuc resolvi potuerit.

Simili autem modo facile intelligitur hanc aequationem quarti gradus

$$\left(\frac{d^4 z}{dy^4}\right) = aa\left(\frac{ddz}{dx^2}\right) + 2ab\left(\frac{dz}{dx}\right) + bbz$$

reduci ad hanc geminatam secundi gradus

$$\left(\frac{ddz}{dy^2}\right) = \pm a\left(\frac{dz}{dx}\right) \pm bz$$

neque difficile est alios casus a posteriori eruere, ubi huiusmodi reductiones ad gradum inferiorem locum inveniunt.

CAPUT III

DE INTEGRATIONE AEQUATIONUM HOMOGENEARUM UBI SINGULI TERMINI FORMULAS DIFFERENTIALES EIUSDEM GRADUS CONTINENT

PROBLEMA 69

416. *Aequationis homogeneae secundi gradus*

$$A\left(\frac{ddz}{dx^2}\right)+B\left(\frac{ddz}{dxdy}\right)+C\left(\frac{ddz}{dy^2}\right)=0$$

integrale seu indolem functionis z *investigare denotantibus litteris* A, B, C *quantitates quascunque constantes.*

SOLUTIO

Hanc aequationem voco homogeneam, quia formulis differentialibus secundi gradus constat neque praeterea alias quantitates variabiles involvit. Ad hanc resolvendam observo ei satisfacere huiusmodi aequationem homogeneam primi gradus

$$\left(\frac{dz}{dx}\right)+\alpha\left(\frac{dz}{dy}\right)=\Delta=\text{Const.};$$

hac enim duplici modo per x et y differentiata oritur

$$\text{I.}\quad\left(\frac{ddz}{dx^2}\right)+\alpha\left(\frac{ddz}{dxdy}\right)=0,$$

$$\text{II.}\quad\left(\frac{ddz}{dxdy}\right)+\alpha\left(\frac{ddz}{dy^2}\right)=0.$$

Iam illa per A, haec vero per $\frac{C}{\alpha}$ multiplicata iunctim propositam producent, si fuerit

$$A\alpha + \frac{C}{\alpha} = B \quad \text{seu} \quad A\alpha\alpha - B\alpha + C = 0,$$

unde duplex valor pro α resultat, quorum uterque per aequationem assumtam dabit partem functionis quaesitae z. Cum igitur sit $\left(\frac{dz}{dx}\right) = \varDelta - \alpha\left(\frac{dz}{dy}\right)$, erit

$$dz = \varDelta dx + (dy - \alpha dx)\left(\frac{dz}{dy}\right);$$

patet $\left(\frac{dz}{dy}\right)$ functionem esse debere ipsius $y - \alpha x$, qua posita $= \Gamma' : (y - \alpha x)$ erit $z = fx + \Gamma : (y - \alpha x)$ denotante f constantem quamcunque.

Quocirca aequationis propositae solutio ita se habebit. Formetur primo aequatio algebraica

$$Auu + Bu + C = 0,$$

cuius factores simplices sint $u + \alpha$ et $u + \beta$, ita ut sit

$$Auu + Bu + C = A(u + \alpha)(u + \beta);$$

tum integrale quaesitum erit

$$z = fx + \Gamma : (y - \alpha x) + \varDelta : (y - \beta x);$$

ubi cum prima pars fx iam in binis functionibus indefinitis contineri sit censenda ob

$$fx = \frac{f(y - \alpha x) - f(y - \beta x)}{\beta - \alpha},$$

succinctius ita exprimetur

$$z = \Gamma : (y - \alpha x) + \varDelta : (y - \beta x),$$

quod ob binas functiones arbitrarias utique pro completo est habendum, unico casu excepto, quo est $\beta = \alpha$. Pro quo casu statuamus $\beta = \alpha + d\alpha$, et cum sit

$$\varDelta : (y - (\alpha + d\alpha)x) = \varDelta : (y - \alpha x) - x d\alpha \varDelta' : (y - \alpha x),$$

quia pars prior iam in membro priori continetur et loco posterioris scribere licet $x\varDelta : (y - \alpha x)$, erit pro casu $\beta = \alpha$ seu $BB = 4AC$ integrale

$$z = \Gamma : (y - \alpha x) + x\varDelta : (y - \alpha x).$$

COROLLARIUM 1

417. Pro casu $\beta = \alpha$ manifestum est integrale etiam hoc modo exprimi posse

$$z = \Gamma : (y - \alpha x) + y\varDelta : (y - \alpha x),$$

quae autem forma ab illa non discrepat.

COROLLARIUM 2

418. Si $C = 0$, ut sit

$$A\left(\frac{ddz}{dx^2}\right) + B\left(\frac{ddz}{dxdy}\right) = 0$$

hincque $Auu + Bu = Au\left(u + \frac{B}{A}\right)$, fit $\alpha = 0$ et $\beta = \frac{B}{A}$ et integrale

$$z = \Gamma : y + \varDelta : \left(y - \frac{B}{A}x\right) = \Gamma : y + \varDelta : (Ay - Bx).$$

Simili modo aequationis

$$B\left(\frac{ddz}{dxdy}\right) + C\left(\frac{ddz}{dy^2}\right) = 0$$

integrale est

$$z = \Gamma : x + \varDelta : (Cx - By).$$

COROLLARIUM 3

419. Porro huius aequationis

$$aa\left(\frac{ddz}{dx^2}\right) + 2ab\left(\frac{ddz}{dxdy}\right) + bb\left(\frac{ddz}{dy^2}\right) = 0$$

ob $aauu + 2abu + bb = aa\left(u + \frac{b}{a}\right)^2$ est integrale

$$z = \Gamma : (ay - bx) + x\varDelta : (ay - bx).$$

SCHOLION

420. Harum integralium forma nulla laborat difficultate, quamdiu aequatio

$$Auu + Bu + C = 0$$

duas habet radices reales, sive sint inaequales sive aequales; quando autem hae radices fiunt imaginariae, ut sit

$$\alpha = \mu + \nu\sqrt{-1} \quad \text{et} \quad \beta = \mu - \nu\sqrt{-1},$$

tum functiones arbitrariae omni fere usu destituuntur. Etsi enim indoles functionum Γ et Δ lineis curvis utcunque ductis repraesentatur, ut $\Gamma:v$ et $\Delta:v$ denotent in iis applicatas abscissae v convenientes, nullo modo patet, quomodo valores

$$\Gamma:(p+q\sqrt{-1}) \quad \text{et} \quad \Delta:(p-q\sqrt{-1})$$

exhiberi debeant, etiamsi imaginaria se mutuo tollant. In quo ingens cernitur discrimen inter functiones continuas et discontinuas, cum in illis semper valores ita expressi

$$\Gamma:(p+q\sqrt{-1})+\Gamma:(p-q\sqrt{-1}) \quad \text{et} \quad \frac{\Delta:(p+q\sqrt{-1})-\Delta:(p-q\sqrt{-1})}{\sqrt{-1}}$$

realiter exhiberi queant, id quod, si Γ et Δ significent functiones discontinuas, nullo modo succedit. His igitur casibus solutio generalis hic inventa ad solas functiones continuas restringenda videtur, quandoquidem discontinuae applicationi et executioni adversantur.

PROBLEMA 70

421. *Proposita hac aequatione tertii gradus homogenea*

$$A\left(\frac{d^3z}{dx^3}\right)+B\left(\frac{d^3z}{dx^2dy}\right)+C\left(\frac{d^3z}{dx\,dy^2}\right)+D\left(\frac{d^3z}{dy^3}\right)=0$$

eius integrale completum invenire.

SOLUTIO

Huic quoque aequationi uti in praecedente problemate satisfacere aequationem differentialem simplicem primi gradus satis luculenter perspicitur, ex quo integrale particulare talem habebit formam

$$z=\Gamma:(y+nx);$$

colligantur hinc singulae formulae differentiales tertii gradus, quae erunt

$$\left(\frac{d^3z}{dx^3}\right)=n^3\Gamma''':(y+nx),\quad \left(\frac{d^3z}{dx^2dy}\right)=n^2\Gamma''':(y+nx),$$
$$\left(\frac{d^3z}{dx\,dy^2}\right)=n\Gamma''':(y+nx),\quad \left(\frac{d^3z}{dy^3}\right)=\Gamma''':(y+nx),$$

quibus substitutis, quoniam divisio per $\Gamma''' : (y + nx)$ succedit, nascitur ista aequatio

$$An^3 + Bn^2 + Cn + D = 0;$$

cuius tres radices si fuerint $n = \alpha$, $n = \beta$, $n = \gamma$, evidens est aequationi propositae satisfacere hanc formam

$$z = \Gamma : (y + \alpha x) + \varDelta : (y + \beta x) + \Sigma : (y + \gamma x);$$

quae cum tres functiones arbitrarias complectatur, dubium non est, quin ea sit integrale completum.

Hoc tantum notetur, si duae radices sint aequales, puta $\gamma = \beta$, integrale fore

$$z = \Gamma : (y + \alpha x) + \varDelta : (y + \beta x) + x\Sigma : (y + \beta x);$$

sin autem adeo omnes tres fuerint inter se aequales, $\gamma = \beta = \alpha$, tum erit integrale quaesitum

$$z = \Gamma : (y + \alpha x) + x\varDelta : (y + \alpha x) + xx\Sigma : (y + \alpha x).$$

Quodsi duae radices fuerint imaginariae, eadem erunt tenenda, quae modo ante sunt observata.

COROLLARIUM 1

422. Ultimus casus, quo tres radices sunt aequales, etiam inde est manifestus, quod, si loco variabilium x et y binae novae $t = x$ et $u = y + \alpha x$ introducantur, aequatio proposita contrahatur in hanc formam $\left(\frac{d^3 z}{dt^3}\right) = 0$, cuius integrale manifesto est

$$z = \Gamma : u + x\varDelta : u + xx\Sigma : u.$$

COROLLARIUM 2

423. Hinc ergo etiam intelligitur, quomodo in aequationibus homogeneis altioris gradus, si aequationes algebraicae inde formatae plures habeant radices aequales, integralia futura sint comparata, ita ut etiam tum neque casus radicum aequalium neque imaginariarum ulli difficultati sit obnoxius.

SCHOLION

424. Casus autem binarum radicum imaginariarum, quibus functiones arbitrariae nullum usum habere videntur, ratione functionum continuarum, quae satisfaciunt, uberiorem evolutionem merentur. Formulae autem his casibus in integrale ingredientes semper ad hanc formam reduci possunt

$$\Gamma : v(\cos.\varphi + \sqrt{-1}\cdot \sin.\varphi) + \Delta : v(\cos.\varphi - \sqrt{-1}\cdot \sin.\varphi),$$

unde primum, si functiones sint potestates, huiusmodi valores colliguntur

$$Av^n \cos. n\varphi + Bv^n \sin. n\varphi \quad \text{seu} \quad Av^n \cos.(n\varphi + \alpha);$$

quotcunque enim huiusmodi valores, constantes A, n et α utcunque mutando, adhiberi possunt. Deinde si functiones denotent logarithmos, prodeunt tales valores

$$Alv + B\varphi.$$

Tertio si functiones sint exponentiales, oriuntur hi

$$e^{v\cos.\varphi}(A\cos.(v\sin.\varphi) + B\sin.(v\sin.\varphi)) = Ae^{v\cos.\varphi}\cos.(v\sin.\varphi + \alpha)$$

et generalius

$$Ae^{v^n\cos.n\varphi}\cos.(v^n\sin.n\varphi + \alpha).$$

Plurimae autem aliae huiusmodi formulae ex doctrina imaginariorum elici possunt, quae utcunque cum his combinatae pro parte integrali ex binis radicibus imaginariis nata usurpari poterunt, unde infinita functionum multitudo nascitur, quae solutionem completam mentiri videtur neque tamen pro completa perinde haberi potest atque usu venit iis casibus, quibus omnes radices sunt reales.

Hic autem observetur nullum adhuc problema mechanicum seu physicum occurrisse, quod ab huiusmodi casu penderet.

PROBLEMA 71

425. *Proposita huiusmodi aequatione homogenea gradus cuiuscunque*

$$A\left(\frac{d^\lambda z}{dx^\lambda}\right) + B\left(\frac{d^\lambda z}{dx^{\lambda-1}dy}\right) + C\left(\frac{d^\lambda z}{dx^{\lambda-2}dy^2}\right) + \text{etc.} = 0$$

eius integrale completum invenire.

SOLUTIO

Formetur hinc aequatio algebraica ordinis λ

$$An^\lambda + Bn^{\lambda-1} + Cn^{\lambda-2} + \text{etc.} = 0,$$

cuius radices numero λ sint

$$n = \alpha, \quad n = \beta, \quad n = \gamma, \quad n = \delta \quad \text{etc.};$$

quae si omnes fuerint inaequales, integrale completum aequationis propositae erit

$$z = \Gamma:(y + \alpha x) + \Delta:(y + \beta x) + \Sigma:(y + \gamma x) + \Theta:(y + \delta x) + \text{etc.},$$

quarum functionum disparium numerus erit $= \lambda$. Sin autem eveniat, ut inter has radices duae pluresve reperiantur aequales, scilicet $\beta = \alpha$, $\gamma = \alpha$ etc., tum functiones has radices aequales involventes respective multiplicari debent per terminos progressionis geometricae huius 1, x, x^2 etc. vel huius 1, y, y^2 etc., ita ut functionum arbitrariarum numerus non minuatur. De radicibus autem imaginariis perpetuo ea sunt notanda, quae ante [§ 420] observavimus, nisi forte functiones arbitrarias formularum imaginariarum excludere nolimus.

COROLLARIUM 1

426. Casu radicum aequalium perinde est, utra serie geometrica utamur, siquidem functiones neque sint ipsius x neque ipsius y tantum. Sin autem hae functiones fuerint vel ipsius x vel ipsius y tantum, tum alterius variabilis diversae progressione geometrica uti oportet.

COROLLARIUM 2

427. Si in aequatione algebraica termini initiales A, B, C etc. evanescant, ut radicum numerus exponente λ minor esse videatur, tum radices deficientes pro infinite magnis sunt habendae, quibus functiones ipsius x tantum respondebunt in integrale introducendae.

COROLLARIUM 3

428. Ita si fuerit $A = 0$, $B = 0$ et $C = 0$, tres radices α, β, γ in infinitum excrescere sunt censendae, ex quibus nascetur pars integralis

$$\Gamma:x + y\Delta:x + y^2\Sigma:x.$$

SCHOLION

429. Quoniam haec pars calculi integralis vix excoli coepit ideoque huius generis investigationes adhuc prorsus sunt reconditae, de hac sectione plura proferre non licet ideoque his partem primam libri secundi, quae in investigatione functionum binarum variabilium ex data quadam differentialium relatione versatur, concludere cogor. Multo autem pauciora circa partem alteram huius libri in medium afferre conceditur, ubi calculus integralis ad functiones trium variabilium accommodatur, hancque ob causam ne operae quidem erit pretium istam partem in sectiones subdividere, multo minus sequentes partes attingere.

CALCVLI INTEGRALIS
LIBER POSTERIOR.

PARS ALTERA
INVESTIGATIO FVNCTIONVM TRIVM VARIABILIVM EX DATA DIFFERENTIALIVM RELATIONE.

CAPUT I

DE FORMULIS DIFFERENTIALIBUS FUNCTIONUM TRES VARIABILES INVOLVENTIUM

PROBLEMA 72

430. *Si v sit functio quaecunque trium quantitatum variabilium x, y et z, eius formulas differentiales primi gradus exhibere.*

SOLUTIO

Cum v sit functio trium variabilium x, y et z, si ea more solito differentietur, eius differentiale in genere ita reperietur expressum

$$dv = pdx + qdy + rdz.$$

Tribus scilicet id constabit partibus, quarum prima pdx seorsim invenitur, si in differentiatione sola quantitas x ut variabilis tractetur binis reliquis y et z ut constantibus spectatis. Simili modo pars secunda qdy impetratur differentiatione functionis v ita instituta, ut sola quantitas y pro variabili, binae reliquae vero x et z pro constantibus habeantur, quod idem de parte tertia rdz est tenendum, quae est differentiale ipsius v variabilitatis solius quantitatis z ratione habita. Hinc patet, quomodo per differentiationem quantitates istae p, q et r seorsim sint inveniendae, quas hic formulas differentiales primi gradus functionis v appellabo, et ne novis litteris in calculum introducendis sit opus, eas naturae suae convenienter ita indicabo

$$p = \left(\frac{dv}{dx}\right), \quad q = \left(\frac{dv}{dy}\right), \quad r = \left(\frac{dv}{dz}\right).$$

Quaelibet ergo functio v trium variabilium x, y et z tres habet formulas differentiales primi gradus ita designandas

$$\left(\frac{dv}{dx}\right), \quad \left(\frac{dv}{dy}\right), \quad \left(\frac{dv}{dz}\right),$$

in quarum qualibet unicae variabilis ratio habetur, dum binae reliquae ut constantes spectantur, et quoniam differentialia per divisionem tolluntur, hae formulae differentiales ad classem quantitatum finitarum sunt referendae.

COROLLARIUM 1

431. Ex tribus formulis differentialibus functionis v inventis eius differentiale solito more sumtum ita conflatur, ut sit

$$dv = dx\left(\frac{dv}{dx}\right) + dy\left(\frac{dv}{dy}\right) + dz\left(\frac{dv}{dz}\right);$$

cuius ergo formae vicissim integrale est ipsa illa functio v vel etiam eadem quantitate quacunque sive aucta sive minuta.

COROLLARIUM 2

432. Si trium variabilium x, y et z functio v fuerit data, eius formulae differentiales singulae

$$\left(\frac{dv}{dx}\right), \quad \left(\frac{dv}{dy}\right), \quad \left(\frac{dv}{dz}\right)$$

iterum erunt functiones certae earundem variabilium x, y et z per differentiationem facile inveniendae. Interim tamen evenire potest, ut una pluresve variabilium ex huiusmodi formulis differentialibus prorsus excedant.

SCHOLION 1

433. Nihil etiam impedit, quominus quantitas v ut functio trium variabilium x, y et z spectari possit, etiamsi forte duas tantum involvat, dum scilicet ratio compositionis ita est comparata, ut tertia quasi casu excesserit, quod eo minus est mirandum, cum idem in functionibus tam unius quam duarum variabilium evenire possit. Quoniam enim functiones unius variabilis commodissime per applicatas cuiuspiam lineae curvae repraesentari solent,

siquidem pro curvae natura applicatae eius ut certae functiones abscissae x spectari possunt, casu, quo linea curva abit in lineam rectam axi parallelam, etsi tum applicata quantitati constanti aequatur, propterea tamen ex illa idea generali, qua ut functio abscissae x spectatur, neutiquam excluditur; neque enim, si quaeratur, qualis sit functio y ipsius x, incongrue is respondere est censendus, qui dicat hanc functionem y aequari quantitati constanti.

Quod deinde ad functiones binarum variabilium x et y attinet, quas semper per intervalla, quibus singula cuiusdam superficiei puncta a quopiam plano distant, repraesentare licet, dum binae variabiles x et y in hoc plano accipiuntur, manifestum est utique superficiem ita comparatam esse posse, ut functio illa vel per solam x vel per solam y determinetur. Quin etiam si superficies fuerit plana ipsique illi plano parallela, functio illa adeo abit in quantitatem constantem neque propterea minus tanquam functio binarum variabilium considerari debebit. Quamobrem etiam quando tractatio circa functiones trium variabilium versatur, in eo genere etiam eiusmodi functiones [continentur], quae tantum vel per binas vel unicam trium variabilium x, y et z determinantur vel adeo ipsae sunt quantitates constantes.

SCHOLION 2

434. In calculo differentiali iam est ostensum functionum plures variabiles involventium differentialia inveniri, si unaquaeque variabilium seorsim tanquam sola esset variabilis spectetur atque omnia differentialia inde nata in unam summam coniiciantur.[1]) Quodsi ergo differentiatio hoc modo instituatur, singulae istae operationes deleto tantum differentiali praebebunt formulas differentiales, quas his signis

$$\left(\frac{dv}{dx}\right),\quad \left(\frac{dv}{dy}\right) \quad \text{et} \quad \left(\frac{dv}{dz}\right)$$

indicamus, simulque intelligitur, quomodo etiam functionum quatuor pluresve variabiles involventium formulae differentiales sint inveniendae. Circa functiones autem trium variabilium x, y et z exempla aliquot subiungamus, quibus earum ternas formulas differentiales exhibebimus.

1) Vide L. Euleri *Institutiones calculi differentialis*, partis prioris § 214, *Leonhardi Euleri Opera omnia*, series I, vol. 10, p. 146. F. E.

EXEMPLUM 1

435. *Si functio trium variabilium sit* $v = \alpha x + \beta y + \gamma z$, *eius formulae differentiales ita se habebunt:*

Cum per differentiationem prodeat $dv = \alpha dx + \beta dy + \gamma dz$, manifestum est fore

$$\left(\frac{dv}{dx}\right) = \alpha, \quad \left(\frac{dv}{dy}\right) = \beta, \quad \left(\frac{dv}{dz}\right) = \gamma$$

sicque omnes tres formulas differentiales esse constantes.

EXEMPLUM 2

436. *Si functio trium variabilium sit* $v = x^\lambda y^\mu z^\nu$, *eius formulae differentiales ita se habebunt:*

Differentiatione more solito peracta fit

$$dv = \lambda x^{\lambda-1} y^\mu z^\nu dx + \mu x^\lambda y^{\mu-1} z^\nu dy + \nu x^\lambda y^\mu z^{\nu-1} dz,$$

unde perspicuum est fore formulas differentiales

$$\left(\frac{dv}{dx}\right) = \lambda x^{\lambda-1} y^\mu z^\nu, \quad \left(\frac{dv}{dy}\right) = \mu x^\lambda y^{\mu-1} z^\nu, \quad \left(\frac{dv}{dz}\right) = \nu x^\lambda y^\mu z^{\nu-1},$$

quae ergo singulae sunt novae functiones omnium trium variabilium x, y, z, nisi exponentes λ, μ, ν sint vel nihilo vel unitati aequales.

EXEMPLUM 3

437. *Si functio* v *duas tantum involvat variabiles* x *et* y *tertia* z *in eius compositionem non ingrediente, formulae differentiales ita se habebunt:*

Quia functio v duas tantum variabiles x et y implicat, eius differentiale huiusmodi formam induet $dv = pdx + qdy + 0dz$, tertia scilicet parte ex variabilitate ipsius z orta evanescente, unde habebimus

$$\left(\frac{dv}{dx}\right) = p, \quad \left(\frac{dv}{dy}\right) = q \quad \text{et} \quad \left(\frac{dv}{dz}\right) = 0.$$

COROLLARIUM

438. Hinc ergo vicissim patet, si fuerit $\left(\frac{dv}{dz}\right) = 0$, tum fore v functionem quamcunque binarum variabilium x et y, quam in posterum ita indicabimus $v = \Gamma:(x, y)$ denotante $\Gamma:(x, y)$ functionem quamcunque binarum variabilium x et y.

SCHOLION

439. Mox ostendemus, quando functio trium variabilium ex data quadam relatione seu conditione formularum differentialium investiganda proponitur, qualibet integratione introduci functionem quamcunque arbitrariam binarum variabilium atque adeo in hoc consistere criterium, quo haec pars calculi integralis a praecedentibus distinguitur. Quemadmodum enim, dum natura functionum unicae variabilis ex data differentialium conditione investigatur, in quo universus liber primus est occupatus, per quamlibet integrationem quantitas constans arbitraria in calculum invehitur, ita in parte praecedente huius secundi libri vidimus, si functiones binarum variabilium ex data formularum differentialium relatione investigari debeant, tum ad essentiam huius tractationis id pertinere, quod qualibet integratione non quantitas constans, sed adeo functio unius variabilis prorsus arbitraria in calculum introducatur; etsi enim plerumque hae functiones veluti $\Gamma:(\alpha x + \beta y)$ ambas variabiles x et y implicabant, tamen ibi tota quantitas $\alpha x + \beta y$ ut unica spectatur, cuius functionem quamcunque illa formula $\Gamma:(\alpha x + \beta y)$ denotat. Nunc igitur, ubi de functionibus trium variabilium agitur, probe notandum est qualibet integratione functionem arbitrariam duarum adeo variabilium in calculum introduci, ex quo simul indolem integrationum, quae circa functiones plurium variabilium versantur, colligere licet.

PROBLEMA 73

440. *Si sit v functio quaecunque trium variabilium x, y et z, eius formulas differentiales secundi altiorumque graduum exhibere.*

SOLUTIO

Cum eius formulae differentiales primi gradus sint tres

$$\left(\frac{dv}{dx}\right), \quad \left(\frac{dv}{dy}\right), \quad \left(\frac{dv}{dz}\right),$$

quaelibet instar novae functionis considerata iterum tres suppeditabit formulas differentiales, quae autem ob

$$\left(\frac{ddv}{dxdy}\right) = \left(\frac{ddv}{dydx}\right)$$

reducentur ad sex sequentes

$$\left(\frac{ddv}{dx^2}\right), \quad \left(\frac{ddv}{dy^2}\right), \quad \left(\frac{ddv}{dz^2}\right), \quad \left(\frac{ddv}{dxdy}\right), \quad \left(\frac{ddv}{dydz}\right), \quad \left(\frac{ddv}{dxdz}\right),$$

ex quarum denominatoribus intelligitur, quaenam trium quantitatum x, y, z in utraque differentiatione pro sola variabili haberi debeat. Simili modo evidens est formulas differentiales tertii gradus dari decem sequentes

$$\begin{array}{llll} \left(\frac{d^3v}{dx^3}\right), & \left(\frac{d^3v}{dx^2dy}\right), & \left(\frac{d^3v}{dxdy^2}\right), & \\ \left(\frac{d^3v}{dy^3}\right), & \left(\frac{d^3v}{dy^2dz}\right), & \left(\frac{d^3v}{dydz^2}\right), & \left(\frac{d^3v}{dxdydz}\right). \\ \left(\frac{d^3v}{dz^3}\right), & \left(\frac{d^3v}{dz^2dx}\right), & \left(\frac{d^3v}{dzdx^2}\right), & \end{array}$$

Formularum porro differentialium quarti gradus numerus est 15, quinti 21 etc., secundum numeros triangulares, simulque ex cuiusque forma perspicuum est, quomodo eius valor ex data functione v per repetitam differentiationem in qualibet unicam variabilem considerando elici debeat.

COROLLARIUM 1

441. En ergo omnes formulas differentiales cuiusque gradus, quas ex qualibet functione trium variabilium derivare licet per differentiationem, quae porro ut functiones trium variabilium spectari possunt.

COROLLARIUM 2

442. Quemadmodum ergo ex huiusmodi functione data omnes eius formulae differentiales ope calculi differentialis inveniuntur, ita vicissim ex data quapiam formula differentiali vel duarum pluriumve relatione quadam ope calculi integralis ipsa illa functio, unde eae nascuntur, investigari debet.

SCHOLION 1

443. In calculo quidem differentiali parum refert, utrum functio differentianda unam pluresve variabiles involvat, cum praecepta differentiandi pro quovis variabilium numero maneant eadem; quam ob causam etiam calculum differentialem secundum hanc functionum varietatem in diversas partes distingui non erat opus. Longe secus autem accidit in calculo integrali, quem secundum hanc functionum varietatem omnino in partes dividi necesse est, quippe quae partes tam ratione propriae indolis quam ratione praeceptorum maxime inter se discrepant. Quemadmodum igitur hanc partem circa functiones trium variabilium occupatam tractari conveniat, exponendum videtur. Ac primo quidem ii casus commodissime evolventur, quibus unius cuiusdam formulae differentialis valor datur, ex quo indolem functionis quaesitae definiri oporteat, quoniam haec investigatio nulla laborat difficultate. Deinde huiusmodi quaestiones aggrediar, quibus relatio quaepiam inter duas pluresve formulas differentiales proponitur; ubi quidem plurimum refert, cuiusnam gradus ea fuerint, siquidem ex primo gradu plures casus expedire licet, dum ex altioribus vix adhuc quicquam in medium afferri potest. Hunc ergo ordinem in ista tractatione observabo.

SCHOLION 2

444. Videri hic posset ad functiones trium variabilium definiendas duas adeo conditiones seu relationes inter formulas differentiales admitti posse neque unica praescripta quaestionem esse determinatam. Quodsi enim ponatur

$$dv = pdx + qdy + rdz,$$

ubi litterae p, q, r vicem gerunt formularum differentialium primi gradus, atque verbi gratia hae duae proponantur conditiones, ut sit $q = p$ et $r = p$ ac propterea

$$dv = p(dx + dy + dz),$$

manifestum est solutionem dari posse, scilicet

$$v = \Gamma : (x + y + z).$$

Verum ad hanc obiectionem respondeo in hoc exemplo casu evenire, ut binae conditiones simul consistere possint; altera enim parumper immutata, ut

manente $q = p$ esse debeat $r = px$ ideoque

$$dv = p(dx + dy + xdz),$$

perspicuum est nullum pro p valorem exhiberi posse, per quem formula differentialis $dx + dy + xdz$ multiplicata integrabilis reddatur, quod unicum exemplum sufficit ad demonstrandum duabus conditionibus praescribendis huiusmodi quaestiones evadere plus quam determinatas neque propterea solutionem admittere nisi certis casibus, quibus quasi altera conditio iam in altera involvitur. Quocirca semper unica relatio inter formulas differentiales proposito omnino sufficit problemati determinando, quod idcirco, quia per integrationem functio arbitraria indefinita ingreditur, aeque parum pro indeterminato est habendum ac problemata calculi integralis communis, quorum solutio constantem arbitrariam introducit.

CAPUT II

DE INVENTIONE FUNCTIONUM TRIUM VARIABILIUM EX DATO CUIUSPIAM FORMULAE DIFFERENTIALIS VALORE

PROBLEMA 74

445. *Dato valore cuiuspiam formulae differentialis primi gradus investigare ipsam functionem trium variabilium, ex qua illa formula differentialis nascitur.*

SOLUTIO

Sit v functio quaesita trium variabilium x, y et z et S earundem functio data quaecunque, cui formula differentialis $\left(\frac{dv}{dx}\right)$ debeat esse aequalis. Cum igitur sit $\left(\frac{dv}{dx}\right) = S$, erit posita sola quantitate x variabili, binis reliquis vero y et z ut constantibus spectatis

$$dv = Sdx \quad \text{ideoque} \quad v = \int Sdx + \text{Const.},$$

ubi notandum est in integratione formulae Sdx ambas quantitates y et z pro constantibus haberi et loco Const. functionem quamcunque ipsarum y et z scribi debere, ex quo functio quaesita ita exhiberi poterit

$$v = \int Sdx + T:(y \text{ et } z);$$

hic scilicet $T:(y$ et $z)$ quantitatem quamcunque ex binis quantitatibus y et z una cum constantibus utcunque conflatam denotat.

Simili modo si proponatur $\left(\frac{dv}{dy}\right) = S$, erit

$$v = \int S dy + T : (x \text{ et } z)$$

et haec aequatio $\left(\frac{dv}{dz}\right) = S$ integrata praebet

$$v = \int S dz + T : (x \text{ et } y).$$

COROLLARIUM 1

446. Hic iam abunde intelligitur integratione huiusmodi functionum loco constantis introduci functionem arbitrariam duarum quantitatum variabilium atque adeo in hoc characterem harum integrationum esse constituendum.

COROLLARIUM 2

447. Hic ergo istud problema solutum dedimus, quo quaeritur functio v trium variabilium x, y, z, ut posito

$$dv = p dx + q dy + r dz$$

fiat vel $p = S$ vel $q = S$ vel $r = S$ existente S functione quacunque data easdem variabiles vel duas vel unicam involvente.

COROLLARIUM 3

448. Quodsi igitur esse debeat $\left(\frac{dv}{dx}\right) = 0$ seu $p = 0$, functio quaesita erit $v = \Gamma : (y \text{ et } z)$, et ut fiat $\left(\frac{dv}{dy}\right) = 0$, erit $v = \Gamma : (x \text{ et } z)$, tum vero ut fiat $\left(\frac{dv}{dz}\right) = 0$, necesse est sit $v = \Gamma : (x \text{ et } y)$.

SCHOLION 1

449. Quemadmodum in praecedente parte functiones arbitrariae unius variabilis per applicatas curvarum quarumcunque, sive regularium sive etiam irregularium, repraesentari poterant, ita in hac parte functiones binarum variabilium arbitrariae per superficiem pro lubitu descriptam repraesentari possunt. Ita si super plano, in quo binae coordinatae x et y more solito assumuntur, superficies quaecunque expansa concipiatur, tertia coordinata distantiam cuiusvis superficiei puncti ab illo plano designans functionem

quamcunque binarum variabilium x et y repraesentabit. Hocque modo aptissime vera idea huiusmodi functionum constitui videtur, cum ex ea non solum ratio harum functionum regularium, sed etiam irregularium perspiciatur.

SCHOLION 2

450. Hic etiam notari convenit huiusmodi functiones binarum variabilium infinitis diversis modis etiam designari posse. Variatis enim in plano memorato binis coordinatis x et y in binas alias t et u, ut sit $t = \alpha x + \beta y$ et $u = \gamma x + \delta y$, manifestum est functionem binarum variabilium t et u seu $\Gamma:(t \text{ et } u)$ convenire cum functione ipsarum x et y seu $\Gamma:(x \text{ et } y)$; si enim loco t et u illi valores pro x et y substituantur, utique prodit functio duas tantum variabiles x et y involvens. Atque multo generalius si t aequetur functioni cuipiam datae ipsarum x et y pariterque u huiusmodi alii functioni, tum $\Gamma:(t \text{ et } u)$ facta substitutione abibit in functionem ipsarum x et y ita exprimendam $\Delta:(x \text{ et } y)$; non enim necesse est, ut idem functionis character Γ rationem compositionis quasi denotans utrinque sit idem, cum hic in genere de functionibus quibuscunque agatur. Quare si in sequentibus forte eiusmodi functiones occurrant $\Gamma:(ax + by \text{ et } fxx + gyy)$ vel $\Gamma:\left(\sqrt{(xx + yy)} \text{ et } l\frac{x}{y}\right)$ etc., earum loco semper haec forma simplex $\Gamma:(x \text{ et } y)$ scribi potest.

SCHOLION 3

451. Solutionis, quam dedimus, consideratio nobis suppeditat sequentes reflexiones. Primo posito

$$dv = pdx + qdy + rdz$$

si debeat esse $p = \left(\frac{dv}{dx}\right) = 0$, fiet

$$dv = qdy + rdz,$$

unde patet v eiusmodi esse quantitatem, cuius differentiale hanc habiturum sit formam $qdy + rdz$; quod fieri nequit, nisi quantitas v fuerit functio binarum variabilium y et z tantum tertia x penitus exclusa; et quia circa quantitates q et r nulla conditio praescribitur, recte pronunciamus loco quantitatis v accipi posse functionem quamcunque binarum variabilium y et z seu esse $v = \Gamma:(y \text{ et } z)$, quam eandem solutionem consideratio formulae $\left(\frac{dv}{dx}\right) = 0$ suggessit.

Deinde si esse debeat generalius $\left(\frac{dv}{dx}\right)=p=S$ denotante S quantitatem quamcunque ex variabilibus x, y, z conflatam, habebimus

$$dv=Sdx+qdy+rdz,$$

quae aequatio ita resolvitur. Quaeratur primo integrale formulae Sdx sola quantitate x ut variabili spectata, quod sit $=V$; haecque quantitas per omnes tres variabiles differentiata praebeat

$$dV=Sdx+Qdy+Rdz;$$

ex quo cum sit $Sdx=dV-Qdy-Rdz$, erit

$$dv=dV+(q-Q)dy+(r-R)dz \quad \text{seu} \quad d.(v-V)=(q-Q)dy+(r-R)dz,$$

unde ut ante patet quantitatem $v-V$ functioni cuicunque binarum variabilium y et z aequari posse. Quare ob $V=\int Sdx$ prodit ut ante

$$v=\int Sdx+\Gamma:(y \text{ et } z);$$

hocque ratiocinium, quo isthuc pervenimus, diligenter notari meretur, cum etiam in parte prima eximium usum praestare possit.

Proposita enim aequatione [§ 296]

$$\left(\frac{ddz}{dy^2}\right)=aa\left(\frac{ddz}{dx^2}\right),$$

quia est

$$d.\left(\frac{dz}{dx}\right)=dx\left(\frac{ddz}{dx^2}\right)+dy\left(\frac{ddz}{dxdy}\right) \quad \text{et} \quad d.\left(\frac{dz}{dy}\right)=dx\left(\frac{ddz}{dxdy}\right)+dy\left(\frac{ddz}{dy^2}\right),$$

erit

$$ad.\left(\frac{dz}{dx}\right)+d.\left(\frac{dz}{dy}\right)=\left(\frac{ddz}{dx^2}\right)(adx+aady)+\left(\frac{ddz}{dxdy}\right)(ady+dx)$$

seu

$$ad.\left(\frac{dz}{dx}\right)+d.\left(\frac{dz}{dy}\right)=(dx+ady)\left(a\left(\frac{ddz}{dx^2}\right)+\left(\frac{ddz}{dxdy}\right)\right),$$

cuius posterioris membri integrale manifesto est $\Gamma:(x+ay)$, hincque

$$\left(\frac{dz}{dy}\right)=-a\left(\frac{dz}{dx}\right)+a\Gamma':(x+ay),$$

quo una integratio absoluta est censenda. Quare cum sit

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right),$$

habebitur

$$dz = \left(\frac{dz}{dx}\right)(dx - ady) + ady\,\Gamma': (x + ay).$$

Sit $\left(\frac{dz}{dx}\right) = p$ et $x - ay = t$, ut fiat

$$dz = pdt + ady\,\Gamma': (t + 2ay)$$

pro duabus variabilibus t et y hincque

$$z = \tfrac{1}{2}\Gamma: (t + 2ay) + \int dt(p - \tfrac{1}{2}\Gamma': (t + 2ay)) = \Gamma: (x + ay) + \varDelta: (x - ay),$$

quia

$$\varDelta: t = \varDelta: (x - ay) \quad \text{et} \quad \Gamma: (t + 2ay) = \Gamma: (x + ay).$$

PROBLEMA 75

452. *Investigare indolem functionis trium variabilium x, y, z, cuius formula quaedam differentialis secundi gradus aequetur datae cuipiam functioni S.*

SOLUTIO

Denotet v functionem quaesitam, et cum eius sex dentur formulae differentiales secundi gradus, ponamus primo esse debere $\left(\frac{ddv}{dx^2}\right) = S$ et integratione semel instituta prodit

$$\left(\frac{dv}{dx}\right) = \int Sdx + \Gamma: (y \text{ et } z)$$

iterumque integrando

$$v = \int dx \int Sdx + x\Gamma: (y \text{ et } z) + \varDelta: (y \text{ et } z),$$

ubi in formulae $\int dx \int Sdx$ duplici integratione sola quantitas x ut variabilis spectatur, quemadmodum iam supra [§ 249] est inculcatum. Similis autem omnino est integratio aequationum

$$\left(\frac{ddv}{dy^2}\right) = S \quad \text{et} \quad \left(\frac{ddv}{dz^2}\right) = S.$$

Pro reliquis formulis differentialibus secundi gradus sufficit hanc unam $\left(\frac{ddv}{dxdy}\right) = S$ resolvisse; quae primo per solam variabilem x integrata dabit

$$\left(\frac{dv}{dy}\right) = \int Sdx + f:(y \text{ et } z).$$

Deinde altera integratione per solam variabilem y instituta colligitur

$$v = \int dy \int Sdx + \int dy f:(y \text{ et } z) + \varDelta:(x \text{ et } z),$$

ubi primum observo partem primam nullo discrimine ordinis inter binas variabiles x et y habito ita $\iint Sdxdy$ exprimi posse. Deinde quaecunque fuerit $f:(y \text{ et } z)$ functio ipsarum y et z, si ea per dy multiplicetur et spectata z ut constante integretur, evidens est denuo functionem ipsarum y et z prodire, et quia illa nullo modo determinatur, etiam hanc fore indeterminatam ideoque arbitrariam, unde statuere poterimus

$$v = \iint Sdxdy + \varGamma:(y \text{ et } z) + \varDelta:(x \text{ et } z).$$

COROLLARIUM 1

453. Hic observo per integrationem formulae $\int dy f:(y \text{ et } z)$ iam sponte formulam $\varDelta:(x \text{ et } z)$ invehi; cum enim ibi sola quantitas y ut variabilis spectetur, loco quantitatis constantis per integrationem adiiciendae functio quaecunque ipsarum x et z scribi poterit.

COROLLARIUM 2

454. Quodsi functio illa data S evanescat, sequentes integrationes provenient:

Si $\left(\frac{ddv}{dx^2}\right) = 0$, erit $v = x\varGamma:(y \text{ et } z) + \varDelta:(y \text{ et } z)$,

si $\left(\frac{ddv}{dy^2}\right) = 0$, erit $v = y\varGamma:(x \text{ et } z) + \varDelta:(x \text{ et } z)$,

si $\left(\frac{ddv}{dz^2}\right) = 0$, erit $v = z\varGamma:(x \text{ et } y) + \varDelta:(x \text{ et } y)$,

si $\left(\frac{ddv}{dxdy}\right) = 0$, erit $v = \varGamma:(x \text{ et } z) + \varDelta:(y \text{ et } z)$,

si $\left(\frac{ddv}{dxdz}\right) = 0$, erit $v = \varGamma:(x \text{ et } y) + \varDelta:(y \text{ et } z)$,

si $\left(\frac{ddv}{dydz}\right) = 0$, erit $v = \varGamma:(x \text{ et } y) + \varDelta:(x \text{ et } z)$.

COROLLARIUM 3

455. Quia hic duplici opus est integratione atque etiam duae functiones arbitrariae, utraque binarum variabilium, in calculum sunt invectae, hoc certissimum est criterium haec integralia inventa esse completa.

SCHOLION

456. Alio etiam modo haec eadem integralia erui possunt, qui nititur principio supra (§ 451) indicato, quodsi fuerit $dv = Sdx + qdy + rdz$, fore

$$v = \int Sdx + f:(y \text{ et } z).$$

Secundum hoc principium ergo, si fuerit $\left(\frac{ddv}{dx^2}\right) = S$, erit

$$d.\left(\frac{dv}{dx}\right) = Sdx + dy\left(\frac{ddv}{dxdy}\right) + dz\left(\frac{ddv}{dxdz}\right),$$

qua forma cum illa collata loco v habemus $\left(\frac{dv}{dx}\right)$ et loco q et r has formulas $\left(\frac{ddv}{dxdy}\right)$ et $\left(\frac{ddv}{dxdz}\right)$, ex quo integrale erit

$$\left(\frac{dv}{dx}\right) = \int Sdx + f:(y \text{ et } z).$$

Cum iam porro sit

$$dv = \left(\frac{dv}{dx}\right)dx + \left(\frac{dv}{dy}\right)dy + \left(\frac{dv}{dz}\right)dz,$$

erit

$$dv = dx\int Sdx + dxf:(y \text{ et } z) + dy\left(\frac{dv}{dy}\right) + dz\left(\frac{dv}{dz}\right),$$

unde pariter manifesto sequitur

$$v = \int dx \int Sdx + x\Gamma:(y \text{ et } z) + \varDelta:(y \text{ et } z).$$

Pari modo operatio est instituenda pro aequatione $\left(\frac{ddv}{dxdy}\right) = S$; inde enim fit

$$d.\left(\frac{dv}{dy}\right) = Sdx + dy\left(\frac{ddv}{dy^2}\right) + dz\left(\frac{ddv}{dydz}\right),$$

cuius integrale est

$$\left(\frac{dv}{dy}\right) = \int Sdx + f:(y \text{ et } z);$$

altera integratio instituatur in hac forma

$$dv = dy\int Sdx + dyf:(y \text{ et } z) + dx\left(\frac{dv}{dx}\right) + dz\left(\frac{dv}{dz}\right),$$

unde ob $\int dyf:(y \text{ et } z) = \Gamma:(y \text{ et } z)$ obtinetur ut ante

$$v = \int\int Sdxdy + \Gamma:(y \text{ et } z) + \varDelta:(x \text{ et } z).$$

PROBLEMA 76

457. *Investigare indolem functionis trium variabilium x, y et z, cuius quaedam formula differentialis tertii gradus aequetur datae cuipiam quantitati S ex illis variabilibus et constantibus utcunque compositae.*

SOLUTIO

Posita functione quaesita $= v$ percurramus non tam singulas eius formulas differentiales tertii gradus quam eas, quarum ratio est diversa.

Sit igitur primo $\left(\frac{d^3 v}{dx^3}\right) = S$ et prima integratio statim dat

$$\left(\frac{ddv}{dx^2}\right) = \int Sdx + 2\Gamma:(y \text{ et } z),$$

tum vero altera

$$\left(\frac{dv}{dx}\right) = \int dx\int Sdx + 2x\Gamma:(y \text{ et } z) + \varDelta:(y \text{ et } z),$$

unde tandem colligitur

$$v = \int dx\int dx\int Sdx + xx\Gamma:(y \text{ et } z) + x\varDelta:(y \text{ et } z) + \Sigma:(y \text{ et } z).$$

Sit secundo $\left(\frac{d^3 v}{dx^2\, dy}\right) = S$ et binae priores integrationes ut ante dant

$$\left(\frac{dv}{dy}\right) = \int dx\int Sdx + x\Gamma:(y \text{ et } z) + \varDelta:(y \text{ et } z);$$

quia nunc, ut vidimus [§ 452], pro $\int dy\Gamma:(y \text{ et } z)$ scribere licet $\Gamma:(y \text{ et } z)$, per tertiam integrationem invenimus

$$v = \int^3 Sdx^2 dy + x\Gamma:(y \text{ et } z) + \varDelta:(y \text{ et } z) + \Sigma:(x \text{ et } z).$$

In his autem duobus casibus omnes formulae differentiales tertii gradus variabilibus permutandis continentur sola excepta ultima hac $\left(\frac{d^3v}{dxdydz}\right)$, quam idcirco seorsim tractari oportet.

Sit igitur $\left(\frac{d^3v}{dxdydz}\right) = S$ et prima integratione per solam variabilem x instituta obtinetur

$$\left(\frac{ddv}{dydz}\right) = \int Sdx + f:(y \text{ et } z);$$

nunc secundo integretur per solam variabilem y ac reperietur

$$\left(\frac{dv}{dz}\right) = \int\int Sdxdy + \Gamma:(y \text{ et } z) + \varDelta:(x \text{ et } z),$$

unde tandem tertia integratio per z dabit

$$v = \int^3 Sdxdydz = \Gamma:(y \text{ et } z) + \varDelta:(x \text{ et } z) + \Sigma:(x \text{ et } y),$$

sicque problema perfecte est resolutum.

COROLLARIUM 1

458. Quoniam hic triplici opus erat integratione, integralia inventa etiam tres functiones arbitrarias complectuntur easque singulas binarum variabilium, quemadmodum natura integralium completorum postulat.

COROLLARIUM 2

459. Si quantitas data S evanescat, integralia haec sequenti modo se habebunt:

Si fuerit $\left(\frac{d^3v}{dx^3}\right) = 0$, erit

$$v = xx\Gamma:(y \text{ et } z) + x\varDelta:(y \text{ et } z) + \Sigma:(y \text{ et } z);$$

si fuerit $\left(\frac{d^3v}{dx^2dy}\right) = 0$, erit

$$v = x\Gamma:(y \text{ et } z) + \varDelta:(y \text{ et } z) + \Sigma:(x \text{ et } z);$$

si fuerit $\left(\frac{d^3v}{dxdydz}\right) = 0$, erit

$$v = \Gamma:(y \text{ et } z) + \varDelta:(x \text{ et } z) + \Sigma:(x \text{ et } y).$$

SCHOLION

460. Eadem integralia etiam altera methodo supra [§ 451] exposita inveniri possunt superfluumque foret singulas operationes hic apponere. Aeque parum autem opus erit has investigationes ad formulas differentiales altiorum graduum prosequi, cum lex progressionis functionum arbitrariarum singulas integralium partes constituentium cum per se tum per ea, quae supra sunt exposita, satis sit manifesta. Quare huic capiti, quo una quaedam formula differentialis quantitati datae aequari debet, plene est satisfactum.

Antequam autem ulterius progredior, duos adhuc casus satis late patentes proponam, quorum resolutio facile ad praecedentes iam tractatas calculi integralis partes reducitur, quam propterea hic tanquam concessam assumere licet, siquidem difficultates, quae in iis occurrunt, non ad praesens institutum sunt referendae.

PROBLEMA 77

461. *Si in relationem propositam, ex qua naturam functionis trium variabilium x, y et z definiri oportet, aliae formulae differentiales non ingrediantur, nisi quae ex unica variabili x oriuntur, quae sunt*

$$\left(\frac{dv}{dx}\right),\quad \left(\frac{ddv}{dx^2}\right),\quad \left(\frac{d^3v}{dx^3}\right)\quad \text{etc.},$$

functionem quaesitam investigare.

SOLUTIO

Cum aequatio propositam continens relationem alias formulas differentiales praeter memoratas non comprehendat, in ea binae quantitates y et z pro constantibus habentur ideoque etiam in singulis integrationibus tanquam tales tractari possunt. Hinc aequatio proposita duas tantum variabiles x et v involvere est censenda et reiectis formularum differentialium vinculis habebitur aequatio differentialis ad librum primum referenda, in qua, si ad altiores gradus exsurgat, elementum dx constans sumtum est putandum. Quodsi ergo praeceptorum ibidem traditorum ope haec aequatio integrari queat, tum loco constantium per singulas integrationes ingressarum substituantur functiones arbitrariae binarum variabilium y et z, veluti $\Gamma:(y \text{ et } z)$, $\Delta:(y \text{ et } z)$ etc., sicque habebitur solutio completa problematis propositi.

COROLLARIUM 1

462. Praeter plurimos igitur integrabilitatis casus in libro I expositos etiam sequentes aequationes differentiales quamtumvis alti gradus resolutionem admittent

$$S = Av + B\left(\frac{dv}{dx}\right) + C\left(\frac{ddv}{dx^2}\right) + D\left(\frac{d^3v}{dx^3}\right) + \text{etc.}$$

et

$$S = Av + Bx\left(\frac{dv}{dx}\right) + Cx^2\left(\frac{ddv}{dx^2}\right) + Dx^3\left(\frac{d^3v}{dx^3}\right) + \text{etc.}$$

COROLLARIUM 2

463. Vinculis enim abiectis eiusmodi habentur aequationes differentiales, quales in extremis capitibus libri I integrare docuimus.[1]) Tantum opus est, ut loco constantium per integrationes ingressarum scribantur tales functiones

$$\Gamma:(y \text{ et } z), \quad \Delta:(y \text{ et } z), \quad \Sigma:(y \text{ et } z) \quad \text{etc.},$$

ut hoc pacto integralia completa obtineantur.

SCHOLION

464. Huc etiam referri possunt eiusmodi relationes propositae, in quibus formulae differentiales bina elementa dx et dy involventes ita continentur, ut hoc dy ubique eundem habeat dimensionum numerum, cuiusmodi sunt

$$\left(\frac{dv}{dy}\right), \quad \left(\frac{ddv}{dx\,dy}\right), \quad \left(\frac{d^3v}{dx^2dy}\right), \quad \left(\frac{d^4v}{dx^3dy}\right) \quad \text{etc.}$$

vel

$$\left(\frac{ddv}{dy^2}\right), \quad \left(\frac{d^3v}{dx\,dy^2}\right), \quad \left(\frac{d^4v}{dx^2dy^2}\right), \quad \left(\frac{d^5v}{dx^3dy^2}\right) \quad \text{etc.},$$

ipsa autem tum quantitas v nusquam occurrat. Si enim tum pro priori casu ponatur $\left(\frac{dv}{dy}\right) = u$, pro posteriori vero $\left(\frac{ddv}{dy^2}\right) = u$, relatio ad casum problematis revocabitur alias formulas differentiales non continens praeter

$$\left(\frac{du}{dx}\right), \quad \left(\frac{ddu}{dx^2}\right), \quad \left(\frac{d^3u}{dx^3}\right) \quad \text{etc.}$$

1) Vide notas p. 296. F. E.

et ipsam forte functionem u. Quare si aequationem per praecepta supra tradita integrare indeque functionem u definire licuerit, tum restituendo loco u vel $\left(\frac{dv}{dy}\right)$ vel $\left(\frac{ddv}{dy^2}\right)$, ut fiat $\left(\frac{dv}{dy}\right) = S$ vel $\left(\frac{ddv}{dy^2}\right) = S$, etiam hinc per praecepta huius capitis ipsa functio v determinabitur. Quin etiam hoc modo resolvi poterunt aequationes huiusmodi tantum formulas differentiales complectentes

$$\left(\frac{d^{\mu+\nu}v}{dy^\mu dz^\nu}\right), \quad \left(\frac{d^{\mu+\nu+1}v}{dx\,dy^\mu dz^\nu}\right), \quad \left(\frac{d^{\mu+\nu+2}v}{dx^2 dy^\mu dz^\nu}\right) \quad \text{etc.},$$

ubi omnia tria elementa dx, dy, dz occurrunt; posito enim $\left(\frac{d^{\mu+\nu}v}{dy^\mu dz^\nu}\right) = u$ tota aequatio alias formulas non continebit praeter

$$\left(\frac{du}{dx}\right), \quad \left(\frac{ddu}{dx^2}\right), \quad \left(\frac{d^3u}{dx^3}\right) \quad \text{etc.}$$

una cum ipsa functione u sicque ad casum huius problematis erit referenda; ex cuius resolutione si prodierit $u = S = \left(\frac{d^{\mu+\nu}v}{dy^\mu dz^\nu}\right)$ existente iam S functione cognita, investigatio ipsius functionis v iam nulla amplius laborat difficultate.

Datur autem praeterea alius casus ad libri II partem priorem reducibilis, quem sequenti problemate sum expediturus.

PROBLEMA 78

465. *Si in relationem propositam, ex qua trium variabilium x, y, z functionem v definiri oportet, aliae formulae differentiales non ingrediuntur, nisi quae ex variabilitate binarum x et y tantum nascuntur tertio elemento dz penitus excluso, functionem v investigare.*

SOLUTIO

Quoniam in aequationem resolvendam, qua relatio proposita continetur, quantitas z non ut variabilis ingreditur, quotcunque integrationes fuerint instituendae, in iis ita quantitas z, tanquam esset constans, tractari debet. Huius ergo aequationis resolutio ad partem praecedentem est referenda, cum functio binarum tantum variabilium x et y ex formularum differentialium relatione data sit investiganda. Quodsi itaque negotium successerit et integrale fuerit inventum, in eo totidem occurrent functiones arbitrariae unius variabilis certo

modo ex x et y conflatae, quot integrationibus fuerit opus; sit $\varGamma: t$ huiusmodi functio, ubi t per x et y dari assumitur, ac nunc, ut ista solutio ad praesens institutum accommodetur, ubi quantitas z variabilibus annumeratur, loco cuiusque functionis arbitrariae $\varGamma: t$ scribatur hic $\varGamma: (t \text{ et } z)$, functio scilicet duarum variabilium, sicque habebitur integrale completum.

COROLLARIUM 1

466. Si ergo haec proposita fuerit aequatio

$$\left(\frac{ddv}{dy^2}\right) = aa\left(\frac{ddv}{dx^2}\right),$$

quia in parte praecedente [§ 296] invenimus

$$v = \varGamma: (x + ay) + \varDelta: (x - ay),$$

pro casu praesente, quo v debet esse functio trium variabilium x, y et z, integrale ita se habebit

$$v = \varGamma: (\overline{x + ay} \text{ et } z) + \varDelta: (\overline{x - ay} \text{ et } z).$$

COROLLARIUM 2

467. Hic scilicet meminisse oportet formam

$$\varGamma: (\overline{x + ay} \text{ et } z)$$

designare functionem quamcunque binarum variabilium, quarum altera sit $= x + ay$, altera vero $= z$; unde ipsam functionem per applicatam ad certam superficiem relatam repraesentare licebit.

SCHOLION

468. Non solum autem aequationes in problemate descriptae ad partem praecedentem calculi integralis reducentur, sed etiam innumerabiles aliae, quae facta quadam substitutione ad eam formam revocantur. Veluti si in aequatione proposita aliae formulae differentiales non occurrant, nisi in quibus omnibus unica dimensio elementi dz reperitur, quae sunt

$$\left(\frac{dv}{dz}\right), \quad \left(\frac{ddv}{dx\,dz}\right), \quad \left(\frac{ddv}{dy\,dz}\right), \quad \left(\frac{d^3v}{dx^2\,dz}\right), \quad \left(\frac{d^3v}{dx\,dy\,dz}\right), \quad \left(\frac{d^3v}{dy^2\,dz}\right) \quad \text{etc.},$$

manifestum est posito $\left(\frac{dv}{dz}\right) = u$ aequationem illam in aliam transformari, ex qua iam functionem u investigari oporteat, eamque ad casum in problemate expositum referri. Quare si inde indoles functionis u definiri potuerit, ut sit $u = S$, restat, ut haec aequatio $\left(\frac{dv}{dz}\right) = S$ resolvatur, unde, ut ante [§ 445] vidimus, fit

$$v = \int S dz + \Gamma : (x \text{ et } y).$$

Hoc idem tenendum est, si aequatio proposita ope substitutionis $\left(\frac{ddv}{dz^2}\right) = u$ vel $\left(\frac{d^3v}{dz^3}\right) = u$ etc. ad casum problematis reduci queat; quin etiam per se est perspicuum, si ope transformationis cuiuscunque aequatio proposita ad casum problematis reduci queat. Tales autem transformationes supra plures exposui, dum vel loco functionis quaesitae v alia u introducitur ponendo $v = Su$ vel ipsae variabiles x, y, z in alias p, q, r mutantur, quae ad illas certam teneant rationem, quod negotium pro casu duarum variabilium supra [§ 232, 240] fusius explicavi; hocque ita perspicuum est, ut similis reductio ad hunc casum trium variabilium facile accommodari queat. In sequentibus tamen forte eiusmodi transformationes occurrent; ad alios ergo casus, ubi omnis generis formulae differentiales occurrunt, progredior vix ultra prima elementa rem producturus.

CAPUT III

DE RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM PRIMI GRADUS

PROBLEMA 79

469. *Si pro functione v trium variabilium x, y, z posito*

$$dv = pdx + qdy + rdz$$

fuerit

$$\alpha p + \beta q + \gamma r = 0,$$

indolem functionis v definire.

SOLUTIO

Cum sit $\gamma dv = \gamma pdx + \gamma qdy - (\alpha p + \beta q)dz$, erit

$$\gamma dv = p(\gamma dx - \alpha dz) + q(\gamma dy - \beta dz)$$

ideoque ponendo $\gamma x - \alpha z = t$ et $\gamma y - \beta z = u$ habebitur $\gamma dv = pdt + qdu$, unde patet quantitatem v aequari functioni cuicunque binarum variabilium t et u, ita ut sit $v = \Gamma:(t \text{ et } u)$ et restitutis valoribus assumtis

$$v = \Gamma:(\overline{\gamma x - \alpha z} \text{ et } \overline{\gamma y - \beta z}),$$

quae ergo est solutio problematis, si inter formulas differentiales proponatur haec conditio, ut sit

$$\alpha\left(\frac{dv}{dx}\right) + \beta\left(\frac{dv}{dy}\right) + \gamma\left(\frac{dv}{dz}\right) = 0,$$

cuius itaque aequationis integrale clarius ita exhibetur

$$v = \Gamma:\left(\overline{\frac{x}{\alpha} - \frac{z}{\gamma}} \text{ et } \overline{\frac{y}{\beta} - \frac{z}{\gamma}}\right).$$

COROLLARIUM 1

470. Evidens est hoc integrale etiam ita exprimi posse

$$v = \Gamma : \left(\overline{\frac{x}{\alpha} - \frac{y}{\beta}} \text{ et } \overline{\frac{y}{\beta} - \frac{z}{\gamma}} \right),$$

quandoquidem in genere, uti supra [§ 450] observavimus, est

$$\Gamma : (x \text{ et } y) = \Delta : (t \text{ et } u),$$

siquidem t et u utcunque per x et y determinentur.

COROLLARIUM 2

471. Quin etiam affirmare licet constitutis his tribus formulis

$$\frac{x}{\alpha} - \frac{y}{\beta}, \quad \frac{y}{\beta} - \frac{z}{\gamma}, \quad \frac{z}{\gamma} - \frac{x}{\alpha}$$

quantitatem v esse functionem quamcunque trium harum formularum; siquidem unaquaeque iam per binas reliquas datur ac propterea v nihilominus functioni duarum tantum quantitatum variabilium aequatur.

PROBLEMA 80

472. *Si posito $dv = pdx + qdy + rdz$ haec conditio requiratur, ut sit*

$$px + qy + rz = nv \quad \textit{seu} \quad nv = x\left(\frac{dv}{dx}\right) + y\left(\frac{dv}{dy}\right) + z\left(\frac{dv}{dz}\right),$$

indolem huius functionis v investigare.

SOLUTIO

Ex conditione praescripta capiatur valor $r = \frac{nv - px - qy}{z}$, quo substituto fit

$$dv - \frac{nvdz}{z} = p\left(dx - \frac{xdz}{z}\right) + q\left(dy - \frac{ydz}{z}\right)$$

seu

$$dv - \frac{nvdz}{z} = pzd.\frac{x}{z} + qzd.\frac{y}{z}.$$

Quo primum membrum integrabile reddatur, multiplicetur per $\frac{1}{z^n}$, ita ut iam habeamus

$$d.\frac{v}{z^n} = \frac{pz}{z^n}\, d.\frac{x}{z} + \frac{qz}{z^n}\, d.\frac{y}{z}.$$

Cum nunc quantitates p et q non sint determinatae, quoniam in genere ex tali aequatione $dV = PdX + QdY$ sequitur $V = \Gamma : (X \text{ et } Y)$, pro nostro casu colligimus

$$\frac{v}{z^n} = \Gamma : \left(\frac{x}{z} \text{ et } \frac{y}{z}\right) \quad \text{seu} \quad v = z^n \Gamma : \left(\frac{x}{z} \text{ et } \frac{y}{z}\right).$$

Si scilicet functio quaecunque binarum quantitatum $\frac{x}{z}$ et $\frac{y}{z}$ per z^n seu etiam, quod eodem redit, per x^n vel y^n multiplicetur, oritur valor idoneus pro functione v conditioni praescriptae satisfaciens.

COROLLARIUM 1

473. Perspicuum autem est formam $\Gamma : \left(\frac{x}{z} \text{ et } \frac{y}{z}\right)$ exprimere eiusmodi functionem, in qua tres variabiles x, y, z ubique constituant nullum dimensionum numerum, ac vicissim omnes huiusmodi functiones in forma illa contineri.

COROLLARIUM 2

474. Multiplicatione autem porro facta per z^n oritur functio homogenea trium variabilium x, y, z, cuius dimensionum numerus est $= n$; unde solutio nostri problematis ita enunciari potest, ut quantitas quaesita v sit functio homogenea trium variabilium x, y et z dimensionum numero existente $= n$.

COROLLARIUM 3

475. Quodsi ergo conditio praescripta sit

$$px + qy + rz = 0 \quad \text{seu} \quad x\left(\frac{dv}{dx}\right) + y\left(\frac{dv}{dy}\right) + z\left(\frac{dv}{dz}\right) = 0,$$

quantitas v erit functio homogenea nullius dimensionis trium variabilium x, y et z.

SCHOLION

476. Simili modo solutio succedit, si conditio praescripta postulet, ut sit

$$\alpha px + \beta qy + \gamma rz = nv \quad \text{seu} \quad \alpha x\left(\frac{dv}{dx}\right) + \beta y\left(\frac{dv}{dy}\right) + \gamma z\left(\frac{dv}{dz}\right) = nv;$$

tum enim ob $r = \frac{nv - \alpha px - \beta qy}{\gamma z}$ fit

$$dv - \frac{nvdz}{\gamma z} = p\left(dx - \frac{\alpha xdz}{\gamma z}\right) + q\left(dy - \frac{\beta ydz}{\gamma z}\right),$$

quae aequatio sequenti forma exhibeatur

$$\frac{\gamma dv}{v} - \frac{ndz}{z} = \frac{px}{v}\left(\frac{\gamma dx}{x} - \frac{\alpha dz}{z}\right) + \frac{qy}{v}\left(\frac{\gamma dy}{y} - \frac{\beta dz}{z}\right),$$

ex qua concludimus integrale primi membri $\gamma lv - nlz$ aequari functioni cuicunque binarum quantitatum $\gamma lx - \alpha lz$ et $\gamma ly - \beta lz$ et logarithmorum numeris sumtis fore

$$\frac{v^\gamma}{z^n} = \Gamma:\left(\frac{x^\gamma}{z^\alpha} \text{ et } \frac{y^\gamma}{z^\beta}\right).$$

Ponamus $\alpha = \frac{1}{\lambda}$, $\beta = \frac{1}{\mu}$ et $\gamma = \frac{1}{\nu}$, ut conditio praescripta sit

$$\frac{px}{\lambda} + \frac{qy}{\mu} + \frac{rz}{\nu} = nv,$$

et solutio reducetur ad hanc formam

$$v = z^{\nu n}\varDelta:\left(\frac{x^\lambda}{z^\nu} \text{ et } \frac{y^\mu}{z^\nu}\right).$$

Quodsi porro scribamus $x^\lambda = X$, $y^\mu = Y$ et $z^\nu = Z$, fiet

$$v = Z^n\varDelta:\left(\frac{X}{Z} \text{ et } \frac{Y}{Z}\right)$$

ideoque quantitas quaesita v est functio homogenea, in qua tres variabiles X, Y et Z ubique eundem dimensionum numerum $= n$ adimplent.

PROBLEMA 81

477. *Si posito* $dv = pdx + qdy + rdz$ *haec conditio praescribatur, ut sit*

$$px + qy + rz = nv + S$$

existente S *functione quacunque data variabilium* x, y, z, *investigare naturam functionis quaesitae* v.

SOLUTIO

Cum conditio praescripta praebeat $r = \frac{nv + S - px - qy}{z}$, erit

$$dv - \frac{nvdz}{z} = \frac{Sdz}{z} + p\left(dx - \frac{xdz}{z}\right) + q\left(dy - \frac{ydz}{z}\right)$$

seu

$$d.\frac{v}{z^n} = \frac{Sdz}{z^{n+1}} + \frac{p}{z^{n-1}}\,d.\frac{x}{z} + \frac{q}{z^{n-1}}\,d.\frac{y}{z}.$$

Sit $x = tz$ et $y = uz$, ut iam S fiat functio trium variabilium t, u et z, et formula differentialis $\frac{Sdz}{z^{n+1}}$ ita integretur, ut quantitates t et u constantes habeantur; quo integrali posito $= V$ erit

$$v = Vz^n + z^n \Gamma : \left(\frac{x}{z} \text{ et } \frac{y}{z}\right),$$

ubi pars posterior significat functionem homogeneam trium variabilium x, y, z numero dimensionum existente $= n$.

COROLLARIUM 1

478. Si S sit quantitas constans $= C$, erit

$$V = \int \frac{Cdz}{z^{n+1}} = -\frac{C}{nz^n}$$

hincque primum integralis membrum

$$Vz^n = -\frac{C}{n},$$

ex quo perspicuum est eundem valorem prodituram fuisse quantitatibus x, y, z inter se permutatis.

COROLLARIUM 2

479. Si S sit functio homogenea ipsarum x, y, z dimensionum numero existente $= m$, quia tum posito $x = tz$ et $y = uz$ sit $S = Mz^m$, ita ut M tantum quantitates t et u involvat ideoque pro constante sit habenda, prodit

$$V = \int M z^{m-n-1} dz = \frac{Mz^{m-n}}{m-n} = \frac{S}{(m-n)z^n}$$

sicque primum integralis membrum erit $= \frac{S}{m-n}$.

COROLLARIUM 3

480. At si hoc casu sit $m = n$, fit $V = Mlz + C = Mlaz$ et primum integralis membrum $= Mz^n laz = Slaz$. Pari iure id autem erit $= Slby$ vel $Slax$, id quod satis est manifestum, cum horum valorum differentia fiat functio homogenea n dimensionum ideoque in altero integralis membro contineatur.

SCHOLION

481. Principium huius solutionis in hoc lemmate latissime patente continetur, quod, si fuerit

$$dV = SdZ + PdX + QdY,$$

ubi S denotat functionem datam, P et Q vero functiones indefinitas, futurum sit

$$V = \int SdZ + \Gamma : (X \text{ et } Y);$$

at hic non sufficit indicasse in integratione formulae SdZ solam quantitatem Z pro variabili haberi, sed insuper notari convenit binas X et Y tanquam constantes tractari debere. Quare si forte S sit proposita functio aliarum trium variabilium x, y, z, ex quibus hae X, Y, Z, quarum ratio hic est habenda, certo modo nascantur, primum loco x, y, z istae X, Y et Z introduci debent, ut fiat S functio harum X, Y et Z; tum vero demum binis X et Y pro constantibus solaque Z pro variabili sumta integrale $\int SdZ$ est capiendum. Ita in casu problematis pro integrali $\int \frac{Sdz}{z^{n+1}}$ quantitates $\frac{x}{z}$ et $\frac{y}{z}$ ut constantes sunt spectandae sola z pro variabili sumta; ex quo in functione S statui oportet $x = tz$ et $y = uz$, ut S fiat functio ipsarum z, $t = \frac{x}{z}$ et $u = \frac{y}{z}$,

quarum binae posteriores pro constantibus sunt habendae. Hoc ergo casu insignis error committeretur, si quis sumta z variabili reliquas x et y ut constantes tractare voluerit, quoniam ambae x et y etiam variabilem z involvere sunt censendae. Quod autem variabilibus permutatis primum integralis membrum idem resultare debeat, ut sit

$$z^n \int \frac{S\,dz}{z^{n+1}} = x^n \int \frac{S\,dx}{x^{n+1}},$$

inde patet, quod posito $x = tz$ et $dx = t\,dz$ ob t constantem sumendam fiat

$$x^n \int \frac{S\,dx}{x^{n+1}} = t^n z^n \int \frac{S\,t\,dz}{t^{n+1} z^{n+1}} = z^n \int \frac{S\,dz}{z^{n+1}};$$

in utraque enim integratione rationes variabilium $\frac{x}{z}$, $\frac{y}{z}$, $\frac{x}{y}$ pro constantibus sunt habendae hincque in reductione facta quantitas $t = \frac{x}{z}$ recte ut constans spectatur.

PROBLEMA 82

482. *Si posito $dv = p\,dx + q\,dy + r\,dz$ haec conditio praescribatur, ut esse debeat*

$$pL + qM + rN = 0$$

existentibus L, M, N functionibus datis respective variabilium x, y et z, nempe L ipsius x, M ipsius y et N ipsius z tantum, naturam functionis quaesitae v definire.

SOLUTIO

Ob $r = \frac{-pL - qM}{N}$ aequatio principalis fit

$$dv = p\left(dx - \frac{L\,dz}{N}\right) + q\left(dy - \frac{M\,dz}{N}\right)$$

vel

$$dv = pL\left(\frac{dx}{L} - \frac{dz}{N}\right) + qM\left(\frac{dy}{M} - \frac{dz}{N}\right).$$

Statuatur

$$t = \int \frac{dx}{L} - \int \frac{dz}{N} \quad \text{et} \quad u = \int \frac{dy}{M} - \int \frac{dz}{N},$$

ut fiat $dv = pLdt + qMdu$, et manifestum est quantitatem v aequari debere functioni cuicunque binarum variabilium t et u, quas ita quoque describere licet, ut positis formulis tribus integralibus

$$\int \frac{dx}{L}, \quad \int \frac{dy}{M} \quad \text{et} \quad \int \frac{dz}{N}$$

pro t et u sumi oporteat differentias inter binas earum.

SCHOLION 1

483. Solutio etiam successisset, dummodo $\frac{L}{N}$ fuisset functio ipsarum x et z et $\frac{M}{N}$ ipsarum y et z tantum; tum enim multiplicatores P et Q ad integrationem apti quaeri debuissent, ut fieret

$$P\left(dx - \frac{Ldz}{N}\right) = dt \quad \text{et} \quad Q\left(dy - \frac{Mdz}{N}\right) = du,$$

et ob $dv = \frac{pdt}{P} + \frac{qdu}{Q}$ foret $v = \Gamma : (t \text{ et } u)$. Permutandis vero variabilibus x, y et z etiam alii casus resolubiles prodeunt. Quando autem quantitates L, M, N aliter sunt comparatae, via non patet certa ad solutionem perveniendi; quae certe haud parum abstrusa videtur, cum pro hoc casu satis simplici

$$(y + z)p + (x + z)q + (x + y)r = 0$$

per plures ambages tandem ad hanc pervenerim solutionem, ut posito

$$t = (x + y + z)(x - z)^2 \quad \text{et} \quad u = (x + y + z)(y - z)^2$$

fiat

$$v = \Gamma : (t \text{ et } u);$$

quoniam igitur binae quantitates t et u, quarum functio quaecunque loco v posita conditioni satisfacit, hoc casu tantopere sunt complicatae, generaliter multo minus solutionem expectare licebit.

SCHOLION 2

484. Ad plures autem alios casus solutio extendi potest. Si functiones datae L, M, N ita fuerint comparatae, ut alias E, F, G, H reperire liceat, quibus fiat

$$E\left(dx-\frac{Ldz}{N}\right)+F\left(dy-\frac{Mdz}{N}\right)=dt$$

et

$$G\left(dx-\frac{Ldz}{N}\right)+H\left(dy-\frac{Mdz}{N}\right)=du,$$

tum enim posito $p=PE+QG$ et $q=PF+QH$ fiet

$$dv=Pdt+Qdu,$$

ubi P et Q sunt functiones indefinitae loco p et q introductae, quantitas v aequabitur functioni cuicunque binarum variabilium t et u seu erit

$$v=\Gamma:(t \text{ et } u).$$

Totum ergo negotium huc redit, ut pro datis functionibus L, M, N functiones E, F et G, H inveniantur, quod quidem semper praestari posse videtur, sed haec ipsa quaestio plerumque difficilior evadit quam ipsa proposita. Sufficit autem binas eiusmodi functiones E et F indeque quantitatem t investigasse, quia deinceps permutandis variabilibus x, y, z una cum respondentibus functionibus L, M, N sponte idoneus valor pro u elicitur. Ita in exemplo ante allato

$$L=y+z,\quad M=x+z,\quad N=x+y$$

postquam invenerimus $t=(x+y+z)(x-z)^2$, sola permutatio statim praebet $u=(x+y+z)(y-z)^2$ vel etiam $u=(x+y+z)(x-y)^2$; perinde enim est, utro valore utamur.

PROBLEMA 83

435. *Si posito* $dv=pdx+qdy+rdz$ *haec conditio praescribatur, ut sit*

$$pqr=1,$$

naturam functionis v *investigare.*

SOLUTIO

Ob $r=\frac{1}{pq}$ erit

$$dv=pdx+qdy+\frac{dz}{pq},$$

unde colligimus

$$v = px + qy + \frac{z}{pq} - \int\left(x\,dp + y\,dq - \frac{z\,dp}{ppq} - \frac{z\,dq}{pqq}\right),$$

qua transformatione id sumus assecuti, ut formula integralis bina tantum differentialia dp et dq involvat. His igitur in locum principalium inductis concludimus illam formulam integralem aequari debere functioni cuicunque binarum variabilium p et q. Sit S talis functio, ut fiat

$$v = px + qy + \frac{z}{pq} - S,$$

et iam superest, ut, cum litterae p et q in calculo retineantur, aliae duae elidantur, id quod inde est petendum, quod sit

$$dS = \left(x - \frac{z}{ppq}\right)dp + \left(y - \frac{z}{pqq}\right)dq$$

ideoque

$$x - \frac{z}{ppq} = \left(\frac{dS}{dp}\right) \quad \text{et} \quad y - \frac{z}{pqq} = \left(\frac{dS}{dq}\right).$$

Nunc igitur solutio ita se habebit. Introductis his ternis variabilibus p, q et z sumtaque binarum p et q functione quacunque S capiatur

$$x = \frac{z}{ppq} + \left(\frac{dS}{dp}\right) \quad \text{et} \quad y = \frac{z}{pqq} + \left(\frac{dS}{dq}\right)$$

ac tum functio quaesita v ita definietur, ut sit

$$v = \frac{3z}{pq} + p\left(\frac{dS}{dp}\right) + q\left(\frac{dS}{dq}\right) - S.$$

Vel si malimus v per ipsas tres variabiles x, y, z exprimere, ex binis aequationibus

$$x = \frac{z}{ppq} + \left(\frac{dS}{dp}\right) \quad \text{et} \quad y = \frac{z}{pqq} + \left(\frac{dS}{dq}\right)$$

quaerantur valores ipsarum p et q, quibus in functione S substitutis erit

$$v = px + qy + \frac{z}{pq} - S$$

sicque quaesito erit satisfactum.

COROLLARIUM 1

486. Si functio S sumatur quantitas constans C, ob $ppq = \frac{z}{x}$ et $pqq = \frac{z}{y}$ erit

$$pq = \sqrt[3]{\frac{zz}{xy}}$$

hincque

$$p = \sqrt[3]{\frac{yz}{xx}} \quad \text{et} \quad q = \sqrt[3]{\frac{xz}{yy}},$$

unde fit

$$v = 3\sqrt[3]{xyz} - C,$$

qui est valor particularis problemati satisfaciens.

COROLLARIUM 2

487. Quoniam in conditione praescripta $pqr = 1$ seu $\left(\frac{dv}{dx}\right)\left(\frac{dv}{dy}\right)\left(\frac{dv}{dz}\right) = 1$ tantum differentialia trium variabilium x, y et z occurrunt, eas quantitatibus constantibus quibusvis augere licet, unde nascitur solutio aliquanto latius patens

$$v = 3\sqrt[3]{(x+a)(y+b)(z+c)} - C.$$

SCHOLION 1

488. Alius datur praeterea casus facilem evolutionem admittens ponendo $S = 2c\sqrt{pq}$, unde colligitur

$$p = \frac{\sqrt{y}}{\sqrt{x}}\sqrt[3]{\frac{z}{\sqrt{xy}-c}} \quad \text{et} \quad q = \frac{\sqrt{x}}{\sqrt{y}}\sqrt[3]{\frac{z}{\sqrt{xy}-c}}$$

ideoque

$$S = 2c\sqrt[3]{\frac{z}{\sqrt{xy}-c}}.$$

Assequimur ergo

$$v = 3\sqrt[3]{z\left(\sqrt{xy}-c\right)^2}$$

et permutandis variabilibus simili modo habebimus

$$v = 3\sqrt[3]{y\left(\sqrt{xz}-b\right)^2} \quad \text{et} \quad v = 3\sqrt[3]{x\left(\sqrt{yz}-a\right)^2},$$

ubi porro pro x, y, z scribere licet $x+f$, $y+g$, $z+h$.

Ceterum patet solutionem generalem perinde succedere, si quantitas r functioni cuicunque ipsarum p et q aequari debeat seu si inter p, q, r aequatio quaecunque proponatur.

SCHOLION 2

489. Quodsi enim posito $dv = pdx + qdy + rdz$ inter ternas formulas

$$p = \left(\frac{dv}{dx}\right), \quad q = \left(\frac{dv}{dy}\right), \quad r = \left(\frac{dv}{dz}\right)$$

aequatio proponatur quaecunque, quae differentiata praebeat

$$Pdp + Qdq + Rdr = 0,$$

tum facto

$$S = \int (xdp + ydq + zdr),$$

ut sit

$$v = px + qy + rz - S,$$

sumatur functio quaecunque trium quantitatum p, q, r, quae sit V, haecque differentiata praebeat

$$dV = Ldp + Mdq + Ndr;$$

tum vero est

$$0 = Pudp + Qudq + Rudr$$

ideoque

$$dV = (L + Pu)dp + (M + Qu)dq + (N + Ru)dr,$$

quae forma ob novam introductam variabilem u latissime patet. Statuatur iam $S = V$ fietque

$$x = L + Pu, \quad y = M + Qu, \quad z = N + Ru,$$

ita ut nunc praeter variabiles p, q, r, quarum una per binas reliquas datur, nova habeatur u, ex quibus iam tres x, y et z ita definivimus, ut per eas vicissim hae p, q, r et u determinentur; tum vero erit

$$v = px + qy + rz - V.$$

Quare pro V sumta quacunque functione trium quantitatum p, q, r, inter quas eiusmodi conditio praescribitur, ut sit

$$Pdp + Qdq + Rdr = 0,$$

sumatur

$$x = Pu + \left(\frac{dV}{dp}\right), \quad y = Qu + \left(\frac{dV}{dq}\right), \quad z = Ru + \left(\frac{dV}{dr}\right)$$

eritque

$$v = (Pp + Qq + Rr)u + p\left(\frac{dV}{dp}\right) + q\left(\frac{dV}{dq}\right) + r\left(\frac{dV}{dr}\right) - V,$$

quae solutio praecedenti ideo est anteferenda, quod in hanc tres quantitates p, q, r aequaliter ingrediuntur.

PROBLEMA 84

490. *Si posito* $dv = pdx + qdy + rdz$ *haec conditio praescribatur, ut esse debeat*

$$pqr = \frac{v^3}{xyz},$$

naturam functionis v *definire.*

SOLUTIO

Ponamus $p = \frac{Pv}{x}$, $q = \frac{Qv}{y}$, $r = \frac{Rv}{z}$ et ob conditionem praescriptam debet esse $PQR = 1$; tum vero erit

$$\frac{dv}{v} = \frac{Pdx}{x} + \frac{Qdy}{y} + \frac{Rdz}{z}.$$

Statuamus nunc

$$lv = V, \quad lx = X, \quad ly = Y, \quad lz = Z$$

et habebimus hanc aequationem

$$dV = PdX + QdY + RdZ,$$

pro qua esse debet $PQR = 1$; quae quaestio cum non discrepet a problemate praecedente, eadem solutio huc quoque facillime transferetur.

SCHOLION

491. Plures casus, quos forte in hoc capite expedire liceat, hic non evolvo, cum quia usus nondum perspicitur, tum vero imprimis, quoniam huius partis calculi integralis prorsus adhuc incognitae prima tantum principia adumbrare constitui. Pro formulis autem differentialibus altiorum graduum, quae in conditionem praescriptam ingrediantur, vix quicquam proferre licet praeter quasdam observationes ad aequationes homogeneas pertinentes, quibus ergo hanc partem calculi integralis sum finiturus simulque toti operi finem impositurus.

CAPUT IV

DE AEQUATIONUM DIFFERENTIALIUM HOMOGENEARUM RESOLUTIONE

PROBLEMA 85

492. *Si v aequetur functioni cuicunque binarum quantitatum t et u ita per tres variabiles x, y et z determinatarum, ut sit*

$$t = \alpha x + \beta z \quad et \quad u = \gamma y + \delta z,$$

eius formulas differentiales omnium graduum inde definire.

SOLUTIO

Cum v sit functio quantitatum

$$t = \alpha x + \beta z \quad \text{et} \quad u = \gamma y + \delta z,$$

eius formulae differentiales ex his duabus variabilibus natae innotescent, scilicet

$$\left(\frac{dv}{dt}\right), \quad \left(\frac{dv}{du}\right), \quad \left(\frac{ddv}{dt^2}\right), \quad \left(\frac{ddv}{dt\,du}\right), \quad \left(\frac{ddv}{du^2}\right) \quad \text{etc.};$$

hinc autem statim colligimus

$$\left(\frac{dv}{dx}\right) = \alpha\left(\frac{dv}{dt}\right), \quad \left(\frac{dv}{dy}\right) = \gamma\left(\frac{dv}{du}\right), \quad \left(\frac{dv}{dz}\right) = \beta\left(\frac{dv}{dt}\right) + \delta\left(\frac{dv}{du}\right),$$

formulas scilicet differentiales primi gradus. Pro formulis autem differentialibus secundi gradus adipiscimur

$$\left(\frac{ddv}{dx^2}\right) = \alpha\alpha\left(\frac{ddv}{dt^2}\right), \quad \left(\frac{ddv}{dy^2}\right) = \gamma\gamma\left(\frac{ddv}{du^2}\right),$$

$$\left(\frac{ddv}{dz^2}\right) = \beta\beta\left(\frac{ddv}{dt^2}\right) + 2\beta\delta\left(\frac{ddv}{dtdu}\right) + \delta\delta\left(\frac{ddv}{du^2}\right),$$

$$\left(\frac{ddv}{dxdy}\right) = \alpha\gamma\left(\frac{ddv}{dtdu}\right), \quad \left(\frac{ddv}{dxdz}\right) = \alpha\beta\left(\frac{ddv}{dt^2}\right) + \alpha\delta\left(\frac{ddv}{dtdu}\right)$$

et

$$\left(\frac{ddv}{dydz}\right) = \beta\gamma\left(\frac{ddv}{dtdu}\right) + \gamma\delta\left(\frac{ddv}{du^2}\right).$$

Simili modo ad tertium gradum ascendimus

$$\left(\frac{d^3v}{dx^3}\right) = \alpha^3\left(\frac{d^3v}{dt^3}\right), \quad \left(\frac{d^3v}{dy^3}\right) = \gamma^3\left(\frac{d^3v}{du^3}\right),$$

$$\left(\frac{d^3v}{dz^3}\right) = \beta^3\left(\frac{d^3v}{dt^3}\right) + 3\beta^2\delta\left(\frac{d^3v}{dt^2du}\right) + 3\beta\delta^2\left(\frac{d^3v}{dtdu^2}\right) + \delta^3\left(\frac{d^3v}{du^3}\right),$$

$$\left(\frac{d^3v}{dx^2dy}\right) = \alpha\alpha\gamma\left(\frac{d^3v}{dt^2du}\right), \quad \left(\frac{d^3v}{dxdy^2}\right) = \alpha\gamma\gamma\left(\frac{d^3v}{dtdu^2}\right),$$

$$\left(\frac{d^3v}{dx^2dz}\right) = \alpha\alpha\beta\left(\frac{d^3v}{dt^3}\right) + \alpha\alpha\delta\left(\frac{d^3v}{dt^2du}\right), \quad \left(\frac{d^3v}{dy^2dz}\right) = \beta\gamma\gamma\left(\frac{d^3v}{dtdu^2}\right) + \gamma\gamma\delta\left(\frac{d^3v}{du^3}\right),$$

$$\left(\frac{d^3v}{dxdz^2}\right) = \alpha\beta\beta\left(\frac{d^3v}{dt^3}\right) + 2\alpha\beta\delta\left(\frac{d^3v}{dt^2du}\right) + \alpha\delta\delta\left(\frac{d^3v}{dtdu^2}\right),$$

$$\left(\frac{d^3v}{dydz^2}\right) = \beta\beta\gamma\left(\frac{d^3v}{dt^2du}\right) + 2\beta\gamma\delta\left(\frac{d^3v}{dtdu^2}\right) + \gamma\delta\delta\left(\frac{d^3v}{du^3}\right),$$

$$\left(\frac{d^3v}{dxdydz}\right) = \alpha\beta\gamma\left(\frac{d^3v}{dt^2du}\right) + \alpha\gamma\delta\left(\frac{d^3v}{dtdu^2}\right),$$

unde facile patet, quomodo has formulas differentiales ad altiores gradus continuari oporteat.

SCHOLION 1

493. Hoc problema fortasse generalius concipi debuisse videbitur quantitates t et u ita per tres variabiles x, y, z definiendo, ut esset

$$t = \alpha x + \beta y + \gamma z \quad \text{et} \quad u = \delta x + \varepsilon y + \zeta z;$$

verum cum haec hypothesis in eum tantum finem sit facta, ut v fieret functio ipsarum t et u, evidens tum quoque v spectari posse ut functionem harum duarum quantitatum $\varepsilon t - \beta u$ et $\delta t - \alpha u$, quarum illa ab y, haec vero ab x erit libera. Quocirca hypothesis assumta latissime patere est censenda; exceptio tamen forte hinc admittenda videbitur, si fuerit $t = x + z$ et $u = x - z$, quia hic ipsius u valor non continetur; verum etiam hoc casu quantitas v ut functio ipsarum $t + u$ et $t - u$ spectata fiet functio ipsarum x et z, qui casus utique in hypothesi continetur sumtis $\beta = 0$ et $\gamma = 0$.

SCHOLION 2

494. Hoc problema ideo praemisi, quia alias aequationes differentiales tractare hic non sustineo, nisi quibus eiusmodi valor satisfacit, ut v aequetur functioni cuicunque binarum novarum variabilium t et u, quae ab principalibus x, y, z ita pendeant, ut sit, quemadmodum assumsi,

$$t = \alpha x + \beta z \quad \text{et} \quad u = \gamma y + \delta z.$$

Huiusmodi autem aequationes, quibus hoc modo satisfieri potest, esse homogeneas facile patet, ita ut aequatio resolvenda constet nonnisi formulis differentialibus eiusdem gradus singulis per constantes quantitates multiplicatis et inter se additis, qua appellatione aequationum homogenearum iam in parte praecedente [§ 416—428] sum usus.

Proposita ergo huiusmodi aequatione homogenea loco singularum formularum differentialium per elementa dx, dy, dz formatarum substituantur valores hic inventi per elementa dt et du formati et tum singula membra, quatenus certam formulam differentialem ex elementis dt et du natam complectuntur, seorsim ad nihilum redigantur indeque rationes $\frac{\beta}{\alpha}$ et $\frac{\delta}{\gamma}$ determinentur, quandoquidem quaestio non tam circa has ipsas quantitates quam earum rationes versatur. Quoniam igitur duae tantum res investigationi relinquuntur, si pluribus aequationibus fuerit satisfaciendum, eiusmodi aequationes homogeneae hac ratione resolvi nequeunt, nisi casu plures illae aequationes ad duas tantum revocentur, id quod in sequentibus clarius explicabitur.

PROBLEMA 86

495. *Proposita aequatione homogenea primi gradus*

$$A\left(\frac{dv}{dx}\right) + B\left(\frac{dv}{dy}\right) + C\left(\frac{dv}{dz}\right) = 0$$

investigare naturam functionis v trium variabilium x, y et z.

SOLUTIO

Fingatur $v = \Gamma:(t \text{ et } u)$ existente

$$t = \alpha x + \beta z \quad \text{et} \quad u = \gamma y + \delta z$$

et facta substitutione ex problemate praecedente aequatio nostra in duas partes dividetur

$$\left(\frac{dv}{dt}\right)(A\alpha + C\beta) + \left(\frac{dv}{du}\right)(B\gamma + C\delta) = 0,$$

quarum utraque seorsim ad nihilum reducta praebet

$$\frac{\beta}{\alpha} = \frac{-A}{C} \quad \text{et} \quad \frac{\delta}{\gamma} = \frac{-B}{C},$$

unde fit

$$t = Cx - Az \quad \text{et} \quad u = Cy - Bz.$$

Quare aequationis propositae integrale completum erit

$$v = \Gamma:(\overline{Cx - Az} \text{ et } Cy - Bz),$$

quod etiam concinnius ita exhiberi potest

$$v = \Gamma:\left(\frac{x}{A} - \frac{z}{C} \text{ et } \frac{y}{B} - \frac{z}{C}\right).$$

COROLLARIUM 1

496. Permutandis variabilibus hoc integrale etiam ita exprimi posse evidens est

$$v = \Gamma:\left(\frac{x}{A} - \frac{y}{B} \text{ et } \frac{y}{B} - \frac{z}{C}\right) \quad \text{vel} \quad v = \Gamma:\left(\frac{x}{A} - \frac{y}{B} \text{ et } \frac{x}{A} - \frac{z}{C}\right),$$

quoniam est

$$\frac{x}{A}-\frac{y}{B}=\left(\frac{x}{A}-\frac{z}{C}\right)-\left(\frac{y}{B}-\frac{z}{C}\right).$$

COROLLARIUM 2

497. Quin etiam constitutis ex aequatione proposita his tribus formulis

$$\frac{x}{A}-\frac{y}{B},\quad \frac{x}{A}-\frac{z}{C},\quad \frac{y}{B}-\frac{z}{C}$$

functio quaecunque ex iis utcunque conflata valorem idoneum pro v suppeditabit. Quoniam enim harum binarum formularum unaquaeque est differentia binarum reliquarum, talis functio duas tantum variabiles complecti est censenda.

COROLLARIUM 3

498. Perinde est, quanam harum trium formarum integralium utamur; quando autem binae novae variabiles t et u inter se fuerint aequales, tum alia est utendum. Veluti si esset $C=0$, prima forma $v=\Gamma:(z \text{ et } z)$, utpote functio solius z, foret inutilis et integrale completum esset futurum

$$v=\Gamma:\left(\frac{x}{A}-\frac{y}{B}\text{ et } z\right)\quad \text{seu}\quad v=\Gamma:(\overline{Bx-Ay}\text{ et } z).$$

PROBLEMA 87

499. *Proposita aequatione homogenea secundi gradus*

$$A\left(\frac{ddv}{dx^2}\right)+B\left(\frac{ddv}{dy^2}\right)+C\left(\frac{ddv}{dz^2}\right)+2D\left(\frac{ddv}{dx\,dy}\right)+2E\left(\frac{ddv}{dx\,dz}\right)+2F\left(\frac{ddv}{dy\,dz}\right)=0$$

casus investigare, quibus eius integrale hac forma $\Gamma:(t$ et $u)$ exprimi potest existente

$$t=\alpha x+\beta z\quad et\quad u=\gamma y+\delta z.$$

SOLUTIO

Facta substitutione secundum formulas in Problemate 85 traditas aequatio proposita in tria membra sequentia resolvetur

$$\left.\begin{aligned}&\left(\frac{ddv}{dt^2}\right)(A\alpha\alpha+C\beta\beta+2E\alpha\beta)\\+&\left(\frac{ddv}{dt\,du}\right)(2C\beta\delta+2D\alpha\gamma+2E\alpha\delta+2F\beta\gamma)\\+&\left(\frac{ddv}{du^2}\right)(B\gamma\gamma+C\delta\delta+2F\gamma\delta)\end{aligned}\right\}=0,$$

quorum singula seorsim nihilo debent aequari. At primum praebet

$$\frac{\beta}{\alpha}=\frac{-E+\sqrt{(EE-AC)}}{C},$$

ultimum vero

$$\frac{\delta}{\gamma}=\frac{-F+\sqrt{(FF-BC)}}{C},$$

qui valores in media, quae ita referatur

$$\frac{C\beta\delta}{\alpha\gamma}+D+\frac{E\delta}{\gamma}+\frac{F\beta}{\alpha}=0,$$

substituti suppeditant hanc aequationem

$$EF-CD=\sqrt{(EE-AC)(FF-BC)},$$

qua aequatione conditio inter coefficientes A, B, C, D, E, F continetur, ut solutio hic applicata locum invenire possit. Haec autem aequatio evoluta dat

$$CCDD-2CDEF+BCEE+ACFF-ABCC=0,$$

unde fit

$$C=\frac{2DEF-BEE-AFF}{DD-AB},$$

quia factor C per multiplicationem est ingressus. Quoties autem haec conditio habet locum, ut sit

$$AFF+BEE+CDD=ABC+2DEF,$$

toties haec expressio algebraica ex aequatione proposita formanda

$$Axx+Byy+Czz+2Dxy+2Exz+2Fyz$$

in duos factores potest resolvi neque ergo aliis casibus solutio hic adhibita locum habere potest.

Quo ergo hos casus solutionem admittentes rite evolvamus, ponamus huius formae factores esse

$$(ax + by + cz)(fx + gy + hz),$$

quod ergo eveniet, si fuerit

$$A = af, \qquad B = bg, \qquad C = ch,$$
$$2D = ag + bf, \quad 2E = ah + cf, \quad 2F = bh + cg,$$

unde utique fit

$$AFF + BEE + CDD = ABC + 2DEF.$$

Hinc autem pro solutione colligitur

$$\text{vel } \frac{\beta}{\alpha} = \frac{-a}{c} \quad \text{vel } \frac{\beta}{\alpha} = \frac{-f}{h}$$

et

$$\text{vel } \frac{\delta}{\gamma} = \frac{-b}{c} \quad \text{vel } \frac{\delta}{\gamma} = \frac{-g}{h},$$

ubi observari oportet pro fractionibus $\frac{\beta}{\alpha}$ et $\frac{\delta}{\gamma}$ valores sibi subscriptos coniungi oportere, ita ut sit

$$\text{vel } t = cx - az \quad \text{et} \quad u = cy - bz,$$
$$\text{vel } t = hx - fz \quad \text{et} \quad u = hy - gz.$$

Quocirca pro his casibus solutionem admittentibus integrale completum erit

$$v = \Gamma : (\overline{cx - az} \text{ et } \overline{cy - bz}) + \Delta : (\overline{hx - fz} \text{ et } \overline{hy - gz})$$

seu

$$v = \Gamma : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) + \Delta : \left(\frac{x}{f} - \frac{z}{h} \text{ et } \frac{y}{g} - \frac{z}{h}\right).$$

COROLLARIUM 1

500. Hoc ergo modo aliae aequationes homogeneae secundi gradus resolvi nequeunt, nisi quae in hac forma continentur

$$af\left(\frac{ddv}{dx^2}\right) + bg\left(\frac{ddv}{dy^2}\right) + ch\left(\frac{ddv}{dz^2}\right)$$
$$+ (ag + bf)\left(\frac{ddv}{dxdy}\right) + (ah + cf)\left(\frac{ddv}{dxdz}\right) + (bh + cg)\left(\frac{ddv}{dydz}\right) = 0;$$

tum vero integrale completum erit

$$v = \Gamma : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) + \Delta : \left(\frac{x}{f} - \frac{z}{h} \text{ et } \frac{y}{g} - \frac{z}{h}\right).$$

COROLLARIUM 2

501. Quo autem facilius dignoscatur, utrum aequatio quaepiam proposita

$$A\left(\frac{ddv}{dx^2}\right) + B\left(\frac{ddv}{dy^2}\right) + C\left(\frac{ddv}{dz^2}\right) + 2D\left(\frac{ddv}{dx\,dy}\right) + 2E\left(\frac{ddv}{dx\,dz}\right) + 2F\left(\frac{ddv}{dy\,dz}\right) = 0$$

eo reduci possit necne, formetur inde haec forma algebraica

$$Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz;$$

quae si resolvi patiatur in duos factores rationales

$$(ax + by + cz)(fx + gy + hz),$$

eius integrale completum hinc statim exhiberi potest.

COROLLARIUM 3

502. Unicus tantum casus, quo duo isti factores inter se fiunt aequales, exceptionem postulat, quoniam tum binae functiones inventae in unam coalescerent. Verum ex superioribus [§ 416] colligitur, si hoc eveniat, ut sit $f = a$, $b = g$ et $h = c$, integrale completum ita exprimi[1])

$$z = x\Gamma : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) + \Delta : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right).$$

SCHOLION 1

503. Quibus ergo casibus aequatio homogenea secundi gradus resolutionem admittit, iis quoque in se complectitur duas aequationes homogeneas primi gradus

$$a\left(\frac{dv}{dx}\right) + b\left(\frac{dv}{dy}\right) + c\left(\frac{dv}{dz}\right) = 0$$

et

$$f\left(\frac{dv}{dx}\right) + g\left(\frac{dv}{dy}\right) + h\left(\frac{dv}{dz}\right) = 0,$$

1) Accuratius forma integralis describitur in § 508. F. E.

quippe quarum utraque illi satisfacit, et harum integralia completa iunctim sumta illius integrale completum suppeditant.

Hinc alia via aperitur aequationum homogenearum secundi gradus integralia inveniendi fingendo aequationem primi gradus ipsis satisfacientem

$$a\left(\frac{dv}{dx}\right)+b\left(\frac{dv}{dy}\right)+c\left(\frac{dv}{dz}\right)=0;$$

tum ex hac per triplicem differentiationem tres novae formentur

$$a\left(\frac{ddv}{dx^2}\right)+b\left(\frac{ddv}{dx\,dy}\right)+c\left(\frac{ddv}{dx\,dz}\right)=0,$$

$$a\left(\frac{ddv}{dx\,dy}\right)+b\left(\frac{ddv}{dy^2}\right)+c\left(\frac{ddv}{dy\,dz}\right)=0,$$

$$a\left(\frac{ddv}{dx\,dz}\right)+b\left(\frac{ddv}{dy\,dz}\right)+c\left(\frac{ddv}{dz^2}\right)=0,$$

quarum prima per f, secunda per g et tertia per h multiplicatae et in unam summam collectae ipsam illam aequationem generalem producunt, cuius integrale supra exhibuimus. Ea ergo quasi productum ex binis aequationibus homogeneis primi gradus spectari poterit, ex quibus coniunctis integrale completum formatur.

SCHOLION 2

504. Infinitae ergo aequationes homogeneae secundi gradus hic excluduntur, quae hoc modo integrationem respuunt seu ad aequationes primi gradus reduci nequeunt; qui casus exclusi omnes ex hoc criterio agnoscuntur, si non fuerit

$$AFF+BEE+CDD=ABC+2DEF.$$

Huius generis est ista aequatio $\left(\frac{ddv}{dx\,dy}\right)=\left(\frac{ddv}{dz^2}\right)$, quae ergo tale integrale, cuiusmodi hic assumsimus, non admittit; neque etiam alia patet via eius integrale completum investigandi. Integralia autem particularia facile innumera exhiberi possunt et quae adeo functiones arbitrarias complectuntur, sed tantum unius quantitatis variabilis, quae in praesenti instituto nonnisi integralia particularia constituere sunt censendae. Si enim ponatur

$$v=\Gamma:(\alpha x+\beta y+\gamma z),$$

facta substitutione fieri debet $\alpha\beta = \gamma\gamma$ seu sumto $\gamma = 1$ debet esse $\alpha\beta = 1$; quare innumerabiles adeo huiusmodi formulae coniunctae satisfaciunt, ut sit

$$v = \Gamma : \left(\frac{\alpha}{\beta}x + \frac{\beta}{\alpha}y + z\right) + \Delta : \left(\frac{\gamma}{\delta}x + \frac{\delta}{\gamma}y + z\right) + \Sigma : \left(\frac{\varepsilon}{\zeta}x + \frac{\zeta}{\varepsilon}y + z\right) + \text{etc.},$$

ubi pro α, β, γ, δ etc. numeros quoscunque accipere licet. Quamvis autem infinitae huiusmodi formulae diversae coniungantur, tamen integrale nonnisi pro particulari haberi potest. Ex quo intelligitur integrationem completam istius aequationis $\left(\frac{ddv}{dxdy}\right) = \left(\frac{ddv}{dz^2}\right)$ maximi esse momenti methodumque eo perveniendi fines Analyseos non mediocriter esse prolaturam.

Aequationes autem homogeneae tertii gradus multo maiorem restrictionem exigunt, ut integratio completa hoc modo succedat, uti sequenti problemate ostendetur.

PROBLEMA 88

505. *Aequationum homogenearum tertii gradus eos casus definire, quibus integrale completum per formam assumtam exhiberi seu ad formam aequationum homogenearum primi gradus reduci potest.*

SOLUTIO

In aequatione homogenea tertii gradus fingatur contineri haec primi gradus

$$a\left(\frac{dv}{dx}\right) + b\left(\frac{dv}{dy}\right) + c\left(\frac{dv}{dz}\right) = 0,$$

quae ut satisfaciat aequationi tertii gradus

$$\begin{aligned} &A\left(\frac{d^3v}{dx^3}\right) + B\left(\frac{d^3v}{dy^3}\right) + C\left(\frac{d^3v}{dz^3}\right) \\ &+ D\left(\frac{d^3v}{dx^2dy}\right) + E\left(\frac{d^3v}{dxdy^2}\right) + F\left(\frac{d^3v}{dx^2dz}\right) + G\left(\frac{d^3v}{dxdz^2}\right) + H\left(\frac{d^3v}{dy^2dz}\right) + I\left(\frac{d^3v}{dydz^2}\right) \\ &+ K\left(\frac{d^3v}{dxdydz}\right) = 0, \end{aligned}$$

necesse est, ut expressio haec algebraica

$$Ax^3 + By^3 + Cz^3 + Dxxy + Exyy + Fxxz + Gxzz + Hyyz + Iyzz + Kxyz$$

factorem habeat $ax + by + cz$; nisi autem alter factor denuo in duos simplices sit resolubilis, ad aequationem homogeneam secundi gradus referetur, quae solutionem respuit. Quare ut integratio completa succedat, necesse est istam expressionem tribus constare factoribus simplicibus, qui sint

$$(ax + by + cz)(fx + gy + hz)(kx + my + nz),$$

hincque aequationis generalis coefficientes ita se habebunt

$$A = afk, \quad B = bgm, \quad C = chn,$$
$$D = afm + agk + bfk, \quad E = agm + bfm + bgk, \quad F = afn + ahk + cfk,$$
$$G = ahn + cfn + chk, \quad H = bgn + bhm + cgm, \quad I = bhn + cgn + chm,$$
$$K = agn + ahm + bfn + bhk + cfm + cgk$$

ac tum integrale completum erit

$$v = \Gamma:\left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) + \varDelta:\left(\frac{x}{f} - \frac{z}{h} \text{ et } \frac{y}{g} - \frac{z}{h}\right) + \Sigma:\left(\frac{x}{k} - \frac{z}{n} \text{ et } \frac{y}{m} - \frac{z}{n}\right),$$

quilibet scilicet factor simplex praebet functionem arbitrariam duarum variabilium.

COROLLARIUM 1

506. In qualibet harum functionum variabiles x, y, z inter se permutare licet; quin etiam quaelibet quasi ex tribus variabilibus conflata spectari potest, prima nempe ex his

$$\frac{x}{a} - \frac{y}{b}, \quad \frac{y}{b} - \frac{z}{c} \quad \text{et} \quad \frac{z}{c} - \frac{x}{a}$$

similique modo de ceteris.

COROLLARIUM 2

507. Si duo factores fuerint aequales, $f = a$, $g = b$, $h = c$, quo casu duae priores functiones in unam coalescerent, earum loco scribi debet

$$x\Gamma:\left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) + \varDelta:\left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right),$$

at si omnes tres fuerint aequales, ut insuper sit $k = a$, $m = b$, $n = c$, integrale completum erit

$$v = xx\Gamma : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right) + x\varDelta : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right)$$
$$+ \Sigma : \left(\frac{x}{a} - \frac{z}{c} \text{ et } \frac{y}{b} - \frac{z}{c}\right).$$

COROLLARIUM 3

508. Quemadmodum hic duas priores partes per xx et x multiplicavimus, ita eas quoque per yy et y, item zz et z multiplicare possemus; perinde enim est, quanam variabili hic utamur, dum ne sit ea, quae forte sola post signum functionis occurrit; scilicet si esset $a = 0$ et functiones quantitatum x et $\frac{y}{b} - \frac{z}{c}$ capi debeant, tum multiplicatores xx et x excludi deberent.

SCHOLION 1

509. Simili modo patet aequationes homogeneas quarti gradus hac methodo resolvi non posse, nisi in quatuor eiusmodi aequationes simplices resolvi et quasi earum producta spectari queant. Etsi enim hic revera nulla resolutio in factores locum habeat, tamen ex allatis exemplis clare perspicitur, quemadmodum ex aequatione differentiali homogenea cuiuscunque gradus expressio algebraica eiusdem gradus ternas variabiles x, y, z involvens debeat formari; quae si in factores simplices formae $ax + by + cz$ resolvi queat, simul inde aequationis differentialis integrale completum facile exhibebitur, cum quilibet factor functionem duarum variabilium suppeditet integralis partem constituentem, ita ut etiam haec pars seorsim sumta aequationi differentiali satisfaciat et pro integrali particulari haberi possit. At si illa expressio algebraica ita fuerit comparata, ut factores quidem habeat simplices, sed non tot, quot dimensiones, singuli quidem integralia particularia praebebunt, quae autem iunctim sumta non integrale completum suppeditabunt. Veluti si proponatur haec aequatio differentialis tertii gradus

$$a\left(\frac{d^3v}{dx^2dy}\right) + b\left(\frac{d^3v}{dx\,dy^2}\right) - a\left(\frac{d^3v}{dx\,dz^2}\right) - b\left(\frac{d^3v}{dy\,dz^2}\right) = 0,$$

quia forma algebraica

$$axxy + bxyy - axzz - byzz$$

factorem habet simplicem $ax+by$, illi utique satisfaciet valor

$$v = \Gamma : \left(\frac{x}{a} - \frac{y}{b} \text{ et } z\right);$$

pro integrali autem completo adhuc desunt duae functiones arbitrariae integrale completum huius aequationis

$$\left(\frac{ddv}{dxdy}\right) - \left(\frac{ddv}{dz^2}\right) = 0$$

continentes, ex qua quippe alter factor $xy - zz$ illius expressionis nascitur. Quoties ergo hae expressiones algebraicae ex aequationibus differentialibus homogeneis altiorum graduum formatae resolutionem in factores, etsi non simplices, admittant, hinc saltem discimus, quomodo earum integratio ad aequationes inferiorum graduum revocari possit, quod in huiusmodi arduis investigationibus sine dubio maximi est momenti.

SCHOLION 2

510. Haec sunt, quae de functionibus trium variabilium ex data quadam differentialium relatione investigandis proferre potui, in quibus utique nonnisi prima elementa huius scientiae continentur, quorum ulterior evolutio sagacitati Geometrarum summo studio est commendanda. Tantum enim abest, ut hae speculationes pro sterilibus sint habendae, ut potius pleraque, quae adhuc in theoria motus fluidorum desiderantur, ad has Analyseos partes sublimiores sint referenda[1]), quarum propterea utilitas neutiquam parti priori calculi integralis postponenda videtur.

Eo magis autem hae partes posteriores excoli merentur, quod theoria fluidorum adeo circa functiones quatuor variabilium versetur, quarum naturam ex aequationibus differentialibus secundi gradus investigari oportet, quam partem ob penuriam materiae ne attingere quidem volui. In hac autem theoria resolutio huius aequationis

$$\left(\frac{ddv}{dt^2}\right) = \left(\frac{ddv}{dx^2}\right) + \left(\frac{ddv}{dy^2}\right) + \left(\frac{ddv}{dz^2}\right)$$

1) Vide L. EULERI Commentationes 227 et 396 (indicis ENESTROEMIANI): *Continuation des recherches sur la théorie du mouvement des fluides,* Mém. de l'acad. d. sc. de Berlin [11] (1755), 1757, p. 316, et *Sectio secunda de principiis motus fluidorum,* Novi comment. acad. sc. Petrop. 14 (1769): I, 1770, p. 270; *LEONHARDI EULERI Opera omnia*, series II, vol. 10 et 11. F. E.

maximi est momenti, ubi litterae x, y, z ternas coordinatas, t vero tempus elapsum exprimunt harumque quatuor variabilium functio quaeritur, quae loco v substituta illi aequationi satisfaciat.[1]) Ex hactenus autem allatis facile colligitur integrale completum huius aequationis duas complecti debere functiones arbitrarias, quarum utraque sit functio trium variabilium, aliasque solutiones omnes minus late patentes pro incompletis esse habendas. Facili autem negotio innumeras solutiones particulares exhibere licet; veluti si ponamus

$$v = \Gamma : (\alpha x + \beta y + \gamma z + \delta t),$$

reperitur

$$\delta\delta = \alpha\alpha + \beta\beta + \gamma\gamma;$$

quod cum infinitis modis fieri possit, infinitae huiusmodi functiones additae valorem idoneum pro v exhibebunt. Deinde etiam satisfaciunt isti valores

$$v = \frac{\Gamma : (t \pm \sqrt{(xx + yy + zz)})}{\sqrt{(xx + yy + zz)}}, \qquad v = \frac{\Gamma : (y \pm \sqrt{(tt - xx - zz)})}{\sqrt{(tt - xx - zz)}},$$

$$v = \frac{\Gamma : (x \pm \sqrt{(tt - yy - zz)})}{\sqrt{(tt - yy - zz)}}, \qquad v = \frac{\Gamma : (z \pm \sqrt{(tt - xx - yy)})}{\sqrt{(tt - xx - yy)}},$$

quorum quidem investigatio iam est difficilior.[2]) Cum autem hae functiones tantum sint unius variabilis, integralia maxime particularia exhibent, quae adeo etiamnum forent particularia, si pro v functiones binarum variabilium haberentur, quales autem ne suspicari quidem licet. Quare cum integrale completum duas adeo functiones arbitrarias trium variabilium complecti debeat, facile intelligitur, quantopere adhuc ab hoc scopo simus remoti.

1) Vide L. EULERI Commentationes 268 et 306 (indicis ENESTROEMIANI): *Lettre de M. EULER à M. DE LA GRANGE . . . Recherches sur la propagation des ébranlemens dans un milieu élastique*, Miscellanea Taurin. 2 (1760/1), 1762, p. 1, [typis denuo expressam:] *Oeuvres de LAGRANGE*, t. XIV, Paris 1892, p. 178, et *Supplément aux recherches sur la propagation du son*, Mém. de l'acad. d. sc. de Berlin [15] (1759), 1766, p. 210, praesertim § 43, p. 236; *LEONHARDI EULERI Opera omnia*, series II, vol. 8 et series III, vol. 1. F. E.

2) Vide L. EULERI Commentationem 307 (indicis ENESTROEMIANI): *Continuation des recherches sur la propagation du son*, Mém. de l'acad. d. sc. de Berlin [15] (1759), 1766, p. 241, praesertim § 33, p. 261; *LEONHARDI EULERI Opera omnia*, series III, vol. 1. F. E.

APPENDIX
DE
CALCVLO VARIATIONVM.

CAPUT I

DE CALCULO VARIATIONUM IN GENERE

DEFINITIO 1

1. *Relatio inter binas variabiles variari dicitur, si valor, quo altera inde per alteram determinatur, incremento infinite parvo augeri concipiatur, quod incrementum variationem eius quantitatis, cui adiicitur, vocabimus.*

EXPLICATIO

2. Primum ergo hic consideratur relatio inter binas variabiles x et y quaecunque aequatione quacunque inter easdem expressa, qua pro singulis valoribus ipsi x tributis valores ipsius y convenientes determinentur; tum vero singuli valores ipsius y particulis infinite parvis utcunque augeri concipiantur, ita ut hi valores variati a veris, quos ex relatione proposita sortiuntur, infinite parum discrepent, atque hoc modo relatio illa inter x et y *variari* dicitur simulque particulae illae infinite parvae valoribus veris ipsius y adiunctae *variationes* appellantur. Imprimis autem hic notandum est has variationes, quibus singuli valores ipsius y augeri concipiuntur, neque inter se statui aequales neque ullo modo a se invicem pendentes, sed ita arbitrio nostro permitti, ut omnes praeter unam vel aliquas certis valoribus ipsius y respondentes plane ut nullas spectare liceat. Nulli scilicet legi hae variationes adstrictae sunt concipiendae neque relatio inter x et y data ullam determinationem in istas variationes inferre est censenda, quas ut prorsus arbitrarias spectare oportet.

COROLLARIUM 1

3. Hinc patet variationes toto coelo differre a differentialibus, etiamsi utraque sint infinite parva ideoque plane evanescant; variatio enim afficit eundem valorem ipsius y eidem valori ipsius x convenientem, dum eius differentiale dy simul sequentem valorem $x + dx$ respicit.

COROLLARIUM 2

4. Si enim ex relatione inter x et y proposita ipsi x conveniat y, ipsi $x + dx$ vero valor ipsius y conveniens ponatur y', tum est $dy = y' - y$; at variatio ipsius y neutiquam pendet a valore sequente y', quin potius utrique y et y' pro lubitu suam variationem seorsim tribuere licet.

SCHOLION

5. Haec variationum idea, quae per se tam nimis vaga quam sterilis videri queat, maxime illustrabitur, si eius originem et quo pacto ad eam est perventum accuratius exposuerimus. Perduxit autem eo potissimum quaestio de curvis inveniendis, quae certa quadam maximi minimive proprietate sint praeditae, unde, ne rem in genere considerando obscuritas offundatur, problema contemplemur, quo linea curva quaeritur, super qua grave delabens e dato puncto citissime ad aliud punctum datum descendat. Atque hic quidem ex natura maximorum et minimorum statim constat curvam ita debere esse comparatam, ut, si eius loco alia curva quaecunque infinite parum ab illa discrepans substituatur, tempus descensus super ea idem prorsus sit futurum. Solutionem ergo ita institui oportet, ut, dum curva quaesita tanquam data spectatur, calculus quoque ad aliam curvam infinite parum ab ea discrepantem accommodetur indeque discrimen, quod in temporis expressionem redundat, supputetur; tum enim hoc ipsum discrimen nihilo aequale positum naturam curvae quaesitae declarabit. Curvae autem istae infinite parum a quaesita discrepantes commodissime ita considerantur, ut applicatae singulis abscissis respondentes particulis infinite parvis vel augeantur vel minuantur, hoc est, ut *variationes* recipere concipiantur. Vulgo quidem sufficit huiusmodi variationem in unica applicata constituisse; nihil autem impedit, quominus pluribus atque adeo omnibus applicatis tales variationes assignentur, cum semper ad eandem solutionem perduci sit necesse. Hoc autem modo non solum vis methodi multo luculentius illustratur, sed etiam inde solutiones

quaestionum huius generis pleniores obtinentur, unde etiam quaestiones ad alias conditiones spectantes enucleare licet. Quam ob causam omnino necessarium videtur, ut calculus variationum in amplissima extensione, cuius quidem est capax, pertractetur.

DEFINITIO 2

6. *Pro data relatione inter binas variabiles quantitates utraque earum variari dicitur, si utraque seorsim incremento infinite parvo augeri concipiatur; unde patet, quomodo intelligendum sit, si utrique variabili sua tribuatur variatio.*

EXPLICATIO

7. Si proposita sit aequatio quaecunque inter binas variabiles x et y, qua earum relatio mutua exprimitur, haec relatio per definitionem duplici modo variari potest, altero, quo manentibus valoribus x singulis y variatio tribuitur, altero vero, quo manentibus valoribus y singuli x variari concipiuntur. Nihil igitur prohibet, quominus utraque variabilis simul suas variationes recipere intelligatur, quas adeo ita capere licet, ut nullo plane nexu inter se cohaereant; duplex ergo hic variatio consideratur, cum in definitione prima unica tantum sit admissa. Rem autem hic ita generaliter contemplamur, ut neutra variatio ulli legi sit adstricta neque etiam variationes ipsius y ullo modo a variationibus ipsius x pendeant.

COROLLARIUM 1

8. Ex casu ergo, quo duplex variatio statuitur, casus prior tanquam species nascitur, si variationes alterius variabilis plane reiiciantur; unde manifestum est casum definitionis secundae in se complecti casum primae.

COROLLARIUM 2

9. Hinc magis elucet, quemadmodum data relatio inter binas variabiles infinitis modis variari possit, simulque intelligitur, quoniam has variationes nulli legi adstrictas assumimus, omnes omnino illius relationis variationes possibiles hac ratione indicari.

SCHOLION 1

10. Variationes quidem alterutri tantum variabili inductae iam omnes variationes possibiles, quae in propositam relationem inter binas variabiles cadere possunt, comprehendunt, ut superfluum videri possit calculum ad duplicem variationem accommodari; verum si indolem rei usumque, cui destinatur, attentius contemplemur, duplicis variationis consideratio neutiquam supervacanea deprehendetur, id quod per Geometriam evidentissime sequentem in modum illustrabitur.

Cum relatio quaecunque inter binas variabiles distinctissime per lineam curvam in plano descriptam repraesentetur, sit AYM (Fig. 1) linea curva aequatione inter coordinatas $AX = x$ et $XY = y$ definita, quae ergo datam illam relationem exhibeat; iam igitur quaelibet linea curva alia Aym ab illa infinite parum discrepans relationem illam variatam repraesentabit, quae, quomodocunque se habeat, semper ita considerari potest, ut eidem abscissae $AX = x$ conveniat applicata variata Xv existente particula Yv eius variatione; quae consideratio quoque pro plerisque circa maxima et minima prolatis quaestionibus sufficit, ubi adeo curva AM in nonnullis tantum elementis variari solet concipi. At si quaestio ita sit comparata, ut inter omnes curvas, quas a dato puncto A ad datam quampiam curvam CD usque ducere licet, ea definiatur AYM, cui maximi minimive proprietas quaedam conveniat, tum eadem proprietas in aliam quamcunque curvam proximam Aym etiam in alio lineae CD puncto m terminatam aeque competere debet sicque pro ultimo curvae quaesitae puncto M tam abscissa AP quam applicata PM variationem recipere est consenda et huiusmodi quidem, quae naturae lineae CD sit consentanea. Quo igitur calculus ad talem variationem ultimo elemento inductam accommodari queat, omnino necesse est, ut pro singulis curvae AM punctis intermediis Y generalissime tam abscissae $AX = x$ quam applicatae $XY = y$ variationes tribuantur quaecunque illiusque variatio statuatur particula Xx, huius vero $= xy - XY$, ex quo indoles simulque usus huiusmodi duplicis variationis clarissime perspicitur.

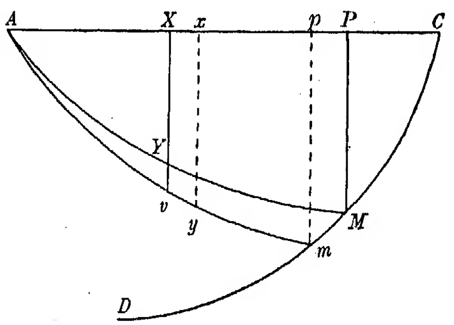

Fig. 1.

SCHOLION 2

11. Quemadmodum consideratio ultimi puncti curvae investigandae nobis hanc insignem dilucidationem suppeditavit, ita etiam subinde primo puncto variationem tribui oportet. Veluti si inter omnes lineas, quas a data quadam curva AB (Fig. 2) ad aliam quandam itidem datam CD ductas concipere licet, ea sit quaerenda, quae maximi minimive cuiuspiam proprietate sit praedita, tum multo magis erit necessarium tam singulis abscissis AX quam applicatis XY variationes quascunque nulla lege adstrictas in calculo assignari, ut deinceps tam ad initii G curvae quaesitae quam eius finis M variationem transferri possint. Quanquam autem haec illustratio ex Geometria est desumta, tamen facile intelligitur ideam variationum inde petitam multo latius patere atque in Analysi absoluta summo usu non esse carituram. Celeberrimus autem DE LA GRANGE, acutissimus Geometra Taurinensis, cui primas speculationes de calculo variationum acceptas referre debemus,[1]) hanc methodum adeo ingeniosissime transtulit ad lineas non continuas veluti ad polygonorum genus referendas, in quo negotio hae duplices variationes ipsi summam praestiterunt utilitatem.

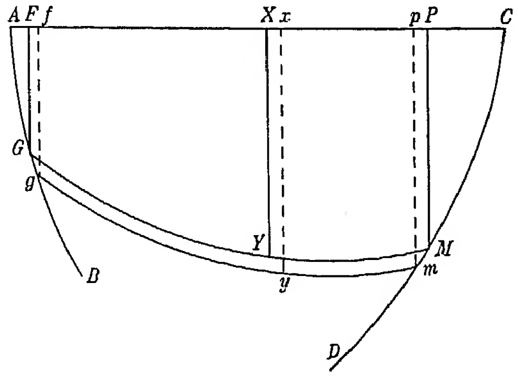

Fig. 2.

1) LAGRANGE methodum suam maxima et minima integralium investigandi publici iuris fecit in Commentatione: *Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies*, Miscellanea Taurin. II_3 (1760/1), 1762, p. 173; *Oeuvres de LAGRANGE*, publiées par les soins de M. I.-A. SERRET, t. I, p. 335. Sed iam die 12. Aug. anni 1755 eam exposuerat in epistola ad EULERUM scripta (*Oeuvres de LAGRANGE*, t. XIV, p. 138), scribens EULERUM ipsum huius generis methodum „*a resolutione geometrica et lineari liberam*" iamdudum desiderasse (L. EULERI *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, Lausannae et Genevae 1744, cap. 2, § 39, *LEONHARDI EULERI Opera omnia*, series I, vol. 24). EULERUS hac communicatione maximo gaudio affectus eximiam viri sagacitatem satis admirari non potuit (vide eius litteras die 6. Sept. 1755 datas, L. EULERI *Opera postuma* 1, Petropoli 1862, p. 555; *Oeuvres de LAGRANGE*, t. XIV, p. 144; *LEONHARDI EULERI Opera omnia*, series III) et explanandae divulgandaeque novae methodo, quam Calculi variationum nomine ornabat, duas Commentationes dedicavit (296 et 297 indicis ENESTROEMIANI): *Elementa calculi variationum* et *Analytica explicatio methodi maximorum et minimorum*, Novi comment. acad. sc. Petrop. 10 (1764), 1766, p 51 et 94; *LEONHARDI EULERI Opera omnia*, series I, vol. 25. F. E.

DEFINITIO 3

12. *Relatio inter tres variabiles duabus aequationibus determinata variari dicitur, si earum vel una vel duae vel omnes tres particulis infinite parvis augeantur, quae earum variationes appellantur.*

EXPLICATIO

13. Cum tres proponantur variabiles quantitates veluti x, y et z, inter quas duae aequationes dari concipiuntur, ex unaquaque earum binas reliquas determinare licet, ita ut tam y quam z tanquam functio ipsius x spectari possit. Hoc autem modo definiri solet linea curva non in eodem plano descripta, dum singula eius puncta per has ternas coordinatas x, y et z more solito assignantur. Quodsi iam talis curva alia quacunque sibi proxima comitetur, ut differentia sit infinite parva, haec nova curva propositae erit variata ac relatio illa inter ternas variabiles x, y, z variata eius naturam exprimere est concipienda. Ex quo, prout bina puncta proxima, alterum in ipsa curva proposita, alterum in variata comitante assumtum, inter se comparantur, fieri potest, ut pro variata vel omnes tres coordinatae prodeant diversae vel duae tantum vel saltem unica, harumque differentiae a coordinatis principalis curvae earum variationes repraesentabunt; quas autem hic ita generalissime contemplari convenit, ut ad omnes omnino curvas proximas extendantur, sive eae per totum tractum a curva proposita fuerint diversae sive tantum in quibusdam portionibus ab ea aberrent, ita ut etiam lineae non continuae, dummodo principali sint proximae, hinc non excludantur. Neque enim hae curvae variatae ulli continuitatis legi sunt adstringendae, ut omnes plane curvas possibiles infinite parum a principali aberrantes in se complectantur.

COROLLARIUM 1

14. Cum puncto ergo quocunque curvae propositae seu principalis comparatur punctum quodpiam curvae variatae infinite parum ab illo dissitum hincque coordinatarum variationes definiri intelliguntur.

COROLLARIUM 2

15. Quia porro ex assumta variabili una x binae reliquae y et z ideoque punctum curvae propositae determinatur, etiam variationes singularum coordinatarum tanquam functiones ipsius x spectare licet, dummodo earum quantitas ut infinite parva spectetur.

COROLLARIUM 3

16. Tres ergo quascunque functiones ipsius x utcunque inter se diversas concipere licet, quae per factores infinite parvos multiplicatae idoneae erunt ad ternas variationes coordinatarum repraesentandas. Quod idem de ternis quibuscunque variabilibus est tenendum, etiamsi non ad Geometriam referantur.

COROLLARIUM 4

17. Simili quoque modo, si relatio tantum inter duas variabiles proponatur, earum variationes tanquam functiones alterius variabilis spectari possunt, modo sint infinite parvae seu, quod eodem redit, per quantitatem infinite parvam multiplicatae.

SCHOLION 1

18. Consideratio autem geometrica maxime est idonea ad has speculationes illustrandas, quae in genere consideratae nimis abstractae atque etiam vagae videri queant. Casus igitur trium variabilium, quarum relatio duabus aequationibus definiri assumitur, luculentissime per curvam non in eodem plano descriptam explicatur, dum illis variabilibus ternae coordinatae designantur. Quodsi enim de huiusmodi curvis quaestio instituatur, ut inter eas definiatur ea, quae maximi minimive proprietate quapiam sit praedita, necesse est, ut eadem proprietas in omnes alias curvas ab ea infinite parum aberrantes aeque competat, id quod ex variationibus debite in calculum introductis est diiudicandum. Cuinam autem usui summa generalitas in variationibus hic stabilita sit futura, inde intelligere licet, si loco duarum curvarum AB et CD (Fig. 2, p. 375) datae sint duae quaecunque superficies, a quarum illa ad hanc eiusmodi lineam curvam duci oporteat, quae maximi minimive quapiam gaudeat proprietate. Tum enim ternarum coordinatarum variationes ita generales considerari oportet, ut curvae quaesitae puncto ad initium in superficiem AB translato variationes ibi ad eandem superficiem accommodari possint idque simili modo in fine ad superficiem CD fieri queat. Ex quo perspicuum est in genere tres variationes in calculum introduci debere, ut eas tam pro initio quam pro fine curvae investigandae ad superficies terminatrices transferre liceat, quippe quarum indoles in utroque termino relationem mutuam inter variationes determinabit.

SCHOLION 2

19. Quemadmodum hic tres variabiles sumus contemplati, quarum relatio duabus aequationibus determinatur, ita etiam calculus variabilium ad quatuor pluresve extendi potest, siquidem relatio per tot aequationes exprimatur, ut per unicam variabilem reliquae omnes determinationem suam nanciscantur, etiamsi huius casus illustratio non amplius ex Geometria tribus tantum dimensionibus inclusa peti queat, nisi forte tempus in subsidium vocare velimus, fluvium continuum a superficie AB ad superficiem CD profluentem, sed temporis lapsu iugiter immutatum considerantes, ita ut tum etiam temporis momentum sit assignandum, quo quaepiam fluvii vena a superficie AB ad superficiem CD porrecta maximi vel minimi proprietate quadam sit praedita. Ad quas variabiles si insuper celeritatis mutabilitatem adiiciamus, haec maiori variationum numero illustrando inservire poterunt. Imprimis autem hinc intelligitur, etiamsi omnes variabiles per unicam determinari assumantur, rationem investigationis tamen ab ea, ubi duae tantum variabiles admittuntur, maxime discrepare, propterea quod singulis suae variationes a reliquis non pendentes tribui debent; neque enim inde, quod inter variabiles ipsas certa quaedam relatio agnoscitur, ideo quoque earum variationes ulli relationi adstrictae sunt censendae; veluti ex casu ante allato manifestum est, ubi curva inter binas superficies AB et CD porrecta et certa maximi minimive proprietate praedita utique ita est in se determinata, ut sumta coordinatarum una binae reliquae determinentur; nihilo vero minus curvae variatae omnes, quae in omnes plagas ab illa deflectere possunt, pro singulis coordinatis recipiunt variationes neutiquam a se invicem pendentes solo initio ac fine excepto, ubi eas ad datas superficies accommodari oportet.

DEFINITIO 4

20. *Relatio inter ternas variabiles unica aequatione definita, ut una earum aequetur functioni binarum reliquarum, variari dicitur, si vel una [vel duae] vel omnes tres illae variabiles particulis infinite parvis augeantur, quae earum variationes vocantur.*

EXPLICATIO

21. Quoniam hic relatio inter ternas variabiles unica aequatione definiri ponitur, duabus pro arbitrio sumtis tertia demum determinatur, ita ut pro

functione duarum variabilium sit habenda. Ea ergo relatione non quaedam linea curva, si rem ad figuras transferre velimus, indicatur, sed tota quaedam superficies, cuius natura aequatione inter ternas coordinatas exprimitur; ex quo intelligitur eadem relatione variata aliam superficiem ab illa infinite parum dissidentem repraesentari, quae variatio ita latissime patere debet, ut variatio vel tantum ad quampiam superficiei portionem restringi vel per totam extendi possit. Prout igitur cum quovis superficiei datae puncto aliud punctum superficiei variatae illi quidem proximum comparatur, fieri potest, ut non solum trium coordinatarum una, sed etiam duae vel adeo omnes tres varientur; unde, quo tractatio in omni amplitudine instituatur, conveniet statim singulis coordinatis suas tribui variationes, quas propterea ita comparatas esse oportet, ut tanquam functiones binarum variabilium spectari possint, cum binis demum determinatis superficiei punctum determinetur.

COROLLARIUM 1

22. Si igitur tres variabiles seu coordinatae sint x, y et z, quemadmodum ex relatione binis x et y pro lubitu valores tribuere licet, unde z valorem determinatum obtineat, itidem variatio ipsius z ab utraque illarum x et y pendere censenda est, quandoquidem, sive alterutra sive ambae mutentur, alia variatio ipsius z resultare debet.

COROLLARIUM 2

23. Quod hic de variatione unius z observatum est, perinde de binis reliquis est intelligendum, ita ut singularum variationes sint tanquam functiones binarum variabilium considerandae; quoniam vero inter x et y et z aequatio datur, perinde est, quarumnam binarum functiones concipiantur, quia functio ipsarum y et z per aequationem ad functionem ipsarum x et y revocari potest, si scilicet loco z suus valor per x et y expressus substituatur.

SCHOLION 1

24. Hac variationum institutione erit utendum, si superficies fuerit investiganda, quae maximi minimive quapiam proprietate sit praedita, quandoquidem calculum tum ita instrui oportet, ut eadem proprietas in superficies illi proximas ac variatas aeque competat. Deinde cum in curvis maximi minimive proprietate praeditis amborum terminorum ratio praescribi soleat,

ut vel in datis punctis vel ad datas lineas curvas vel adeo superficies terminentur, similis conditio hic est admittenda, ut superficies quaerenda circumquaque definiatur vel data quadam superficie circumscribatur; cuius posterioris conditionis ut ratio haberi possit, omnino necesse est, ut omnibus tribus coordinatis variationes generalissimae a se invicem neutiquam pendentes tribuantur, quo eae deinceps in extrema ora ad naturam superficiei terminantis accommodari queant. Hic quidem fatendum est methodum maximorum et minimorum vix adhuc ad huiusmodi investigationes esse promotam tantasque difficultates hic occurrere, ad quas superandas multo maiora Analyseos incrementa requiri videntur. Verum ob hanc ipsam causam eo magis erit enitendum, ut principia huius methodi, quae calculo variationum continentur, solide stabiliantur simulque clare ac distincte proponantur.

SCHOLION 2

25. Vix opus esse arbitror hic animadvertere istum calculum simili modo ad plures tribus variabiles amplificari posse, etiamsi quaestiones geometricae non amplius dilucidationem suppeditent; ipsa enim Analysis non uti Geometria certo dimensionum numero limitari est censenda. Quando autem plures variabiles considerantur, ante omnia perpendi convenit, utrum earum relatio mutua unica tantum aequatione exprimatur an pluribus; quae tot esse possunt, ut multitudo unitate tantum a numero variabilium deficiat, quo casu omnes tanquam functiones unius spectare licet. Sin autem paucioribus aequationibus constet relatio, singulae variabiles erunt functiones duarum pluriumve variabilium et quolibet quoque casu variationes singulis tributae tanquam functiones totidem variabilium tractari debent, siquidem hunc calculum generalissime expedire velimus.

DEFINITIO 5

26. *Calculus variationum est methodus inveniendi variationem, quam recipit expressio ex quotcunque variabilibus utcunque conflata, dum variabilibus vel omnibus vel aliquibus variationes tribuuntur.*

EXPLICATIO

27. In hac definitione nulla fit mentio relationis, quam hactenus inter variabiles dari assumsimus; cum enim hic calculus potissimum in hac ipsa

relatione investiganda sit occupatus, quae scilicet maximi minimive proprietate sit praedita, quamdiu ea adhuc est incognita, eius rationem in calculo neutiquam habere licet, sed potius eum ita tractari convenit, quasi variabiles nulla plane relatione inter se essent connexae. Calculum igitur ita instrui convenit, ut, si singulis variabilibus, quae in calculum ingrediuntur, variationes tribuantur quaecunque, omnis generis expressionum, quae utcunque ex iis fuerint conflatae, variationes inde oriundae investigari doceantur; quibus in genere inventis tum demum eiusmodi quaestiones evolvendae occurrunt, qualem relationem inter variabiles statui oporteat, ut variatio illa inventa sit vel nulla, uti in investigatione maximorum seu minimorum usu venit, vel alio certo quodam modo sit comparata, prout natura quaestionum exegerit. Hoc modo si istius calculi praecepta tradantur, nihil impedit, quominus etiam eiusmodi quaestiones tractentur, in quibus statim relatio quaedam inter variabiles tanquam data assumitur ac certae cuiusdam expressionis ex iis formatae variatio ex variabilium variationibus nata desideratur. Ex quo intelligitur hunc calculum ad quaestiones plurimas diversissimi generis accommodari posse.

COROLLARIUM 1

28. Quaestiones ergo in hoc calculo tractandae huc redeunt, ut proposita expressione quacunque ex quotcunque variabilibus utcunque conflata eius incrementum definiatur, si singulae variabiles suis variationibus augeantur.

COROLLARIUM 2

29. Similis igitur omnino est calculus variationum calculo differentiali, dum in utroque variabilibus incrementa infinite parva tribuuntur. Quatenus autem, uti iam observavimus [§ 3, 4], variationes a differentialibus discrepant adeoque simul cum iis consistere possunt, eatenus summum discrimen inter utrumque calculum est agnoscendum.

SCHOLION

30. Ex observationibus supra allatis discrimen hoc maxime fit manifestum; ubi enim calculus refertur ad lineam curvam, quam cum alia sibi proxima comparari oportet, per differentialia a puncto quovis curvae ad alia puncta eiusdem curvae progredimur; quando autem ab hac curva ad alteram sibi proximam transilimus, transitus, quatenus est infinite parvus, fit per

variationes. Idem evenit in superficiebus ad alias sibi proximas relatis, ubi differentialia in eadem superficie concipiuntur, variationibus vero ab una in alteram transilitur. Eadem omnino est ratio, si res analytice consideretur sine ullo respectu ad figuras geometricas, ubi semper variationes quantitatum variabilium a suis differentialibus sollicite distingui oportet, quem in finem variationes signo diverso indicari conveniet.

HYPOTHESIS

31. *Variationem cuiusque quantitatis variabilis littera δ eidem quantitati praefixa in posterum designabimus, ita ut δx, δy, δz designent variationes quantitatum x, y, z, ac si V fuerit expressio quaecunque ex iis conflata, eius variatio hoc modo δV nobis indicabitur.*

COROLLARIUM 1

32. Significat ergo δx incrementum illud infinite parvum, quo quantitas x augeri concipitur, ut eiusdem valor variatus prodeat, ex quo vicissim intelligitur valorem variatum ipsius x fore $x + \delta x$.

COROLLARIUM 2

33. Quatenus ergo expressio V ex variabilibus x, y et z conflatur, si earum loco scribantur valores variati $x + \delta x$, $y + \delta y$ et $z + \delta z$ atque a valore hoc modo pro V resultante subtrahatur ipsa V, residuum erit variatio δV.

COROLLARIUM 3

34. Hactenus ergo omnia perinde se habent atque in calculo differentiali, ac si V fuerit functio quaecunque ipsarum x, y et z, sumto eius differentiali more solito tantum ubique loco d scribatur δ et habebitur eius variatio δV.

SCHOLION 1

35. Quoties ergo V est functio quaecunque quantitatum variabilium x, y, z, eius variatio iisdem regulis inde elicitur ac differentiale eius, ex quo calculus variationum prorsus cum calculo differentiali congruere videri posset,

cum sola signi diversitas levis sit momenti. Verum probe perpendendum est hic non omnes quantitates, quarum variationes requiruntur, in genere functionum comprehendi posse; quamobrem etiam in definitione [§ 26] vocabulo *expressionis* sum usus, cui longe ampliorem significatum attribuo. Quatenus enim ad relationem mutuam variabilium respicere non licet, quia est incognita, eatenus eiusmodi expressiones seu formulae, in quas variabilium differentialia atque etiam integralia ingrediuntur, non amplius tanquam merae functiones variabilium spectari possunt ac formularum tam differentialium quam integralium variatio peculiaria praecepta postulat; sicque totum negotium huc redit, ut, quemadmodum formularum utriusque generis variationes investigari conveniat, doceamus, ex quo tractatio nostra evadit bipartita.

SCHOLION 2

36. In ipsa autem tractatione maximum exoritur discrimen ex numero variabilium, qui si binarium superet, vix adhuc perspicitur, quomodo calculus sit expediendus. Cum enim pluribus introductis variabilibus etiam differentialium consideratio longe aliter expendatur, dum plerumque binarum tantum differentialia ita inter se comparari solent, quasi reliquae variabiles manerent constantes, similis quoque ratio in variationibus erit habenda, in quo etiamnunc tantae difficultates occurrunt, ut vix pateat, quomodo eas superare liceat; ante omnia certe prima huius calculi principia accuratissime evolvi erit necesse, ut ex intima rei natura calculi praecepta repetantur, in quo plerumque summae difficultates offendi solent. Primum igitur hunc calculum ad duas tantum variabiles accommodatum, quemadmodum is quidem adhuc tractari est solitus, explicare conabor variationes tam formularum differentialium quam integralium investigaturus; tum vero, si quid lucis ex ipsa hac tractatione affulserit, quoque ad tres pluresve variabiles contemplandas progrediar.

CAPUT II

DE VARIATIONE FORMULARUM DIFFERENTIALIUM DUAS VARIABILES INVOLVENTIUM

THEOREMA 1

37. *Variatio differentialis semper aequalis est differentiali variationis seu est* $\delta d V = d \delta V$, *quaecunque fuerit quantitas* V, *quae, dum per differentialia crescit, etiam variationem recipit.*

DEMONSTRATIO

Quantitas variabilis V spectari potest tanquam applicata curvae cuiuspiam, quae suis differentialibus per eandem curvam progrediatur, suis variationibus vero in aliam curvam illi proximam transiliat. Dum autem in eiusdem curvae punctum proximum promovetur, fit eius valor $= V + dV$, qui sit $= V'$, ideoque $dV = V' - V$; ex quo variatio ipsius dV, hoc est $\delta d V$, erit $= \delta V' - \delta V$. Verum $\delta V'$ est valor proximus, in quem δV suo differentiali auctum abit, ita ut sit $\delta V' = \delta V + d \delta V$ seu $\delta V' - \delta V = d \delta V$, unde evidens est fore $\delta d V = d \delta V$ seu variationem differentialis esse aequalem differentiali variationis, prorsus uti theorema affirmat.

COROLLARIUM 1

38. Hinc variatio differentialis secundi ddV ita definitur, ut sit

$$\delta d d V = d \delta . d V,$$

at cum sit $\delta d V = d \delta V$, aequalitas erit inter has formulas

$$\delta d d V = d \delta d V = d d \delta V.$$

COROLLARIUM 2

39. Eodem modo pro differentialibus tertii ordinis erit

$$\delta d^3 V = d\delta dd V = dd\delta d V = d^3 \delta V$$

et pro differentialibus quarti ordinis variatio ita se habebit, ut sit

$$\delta d^4 V = d\delta d^3 V = dd\delta dd V = d^3 \delta d V = d^4 \delta V,$$

similique modo pro altioribus gradibus.

COROLLARIUM 3

40. Si igitur variatio desideretur differentialis cuiuscunque gradus, signum variationis δ, ubicunque libuerit, inter signa differentiationis d inseri potest; in ultimo autem loco positum declarat variationem differentialis cuiusvis gradus aequalem esse differentiali eiusdem gradus ipsius variationis.

COROLLARIUM 4

41. Cum igitur sit $\delta d^n V = d^n \delta V$, res semper eo reducitur, ut variationis quantitatis V seu ipsius δV differentialia cuiusque gradus capi possint, atque in hac reductione praecipua vis huius novi calculi est constituenda.

SCHOLION 1

42. Vis demonstrationis in hoc potissimum est sita, quod δV abeat in $\delta V'$, si quantitas V suo differentiali increscit, quod quidem ex natura differentialium per se est manifestum; interim tamen iuvabit id per Geometriam illustrasse. Pro curva quacunque EF (Fig. 3) sint coordinatae $AX = x$ et $XY = y$; in qua si per intervallum infinite parvum YY' progrediamur, erit in differentialibus

$$AX' = x + dx \quad \text{et} \quad X'Y' = y + dy$$

ideoque

$$dx = AX' - AX \quad \text{et} \quad dy = X'Y' - XY.$$

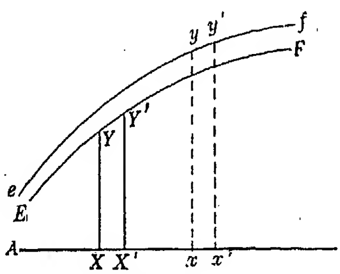

Fig. 3.

Nunc concipiamus aliam curvam ef illi proximam, cuius puncta y et y' cum illius punctis Y et Y' comparentur, ad quae propterea per variationes

transitus fiat, ac sumtis simili modo coordinatis erit

$$Ax = x + \delta x \quad \text{et} \quad xy = y + \delta y$$

ideoque

$$\delta x = Ax - AX \quad \text{et} \quad \delta y = xy - XY;$$

tum vero erit

$$Ax' = x + dx + \delta(x + dx) \quad \text{et} \quad x'y' = y + dy + \delta(y + dy),$$

quatenus a puncto Y' per variationem in punctum y' transilimus. Verum ad idem punctum y' quoque ex puncto y per differentiationem pervenimus, unde colligitur

$$Ax' = x + \delta x + d(x + \delta x) \quad \text{et} \quad x'y' = y + \delta y + d(y + \delta y).$$

His iam valoribus cum illis collatis prodit

$$x + dx + \delta x + \delta dx = x + \delta x + dx + d\delta x$$

et

$$y + dy + \delta y + \delta dy = y + \delta y + dy + d\delta y,$$

unde manifesto sequitur fore

$$\delta dx = d\delta x \quad \text{et} \quad \delta dy = d\delta y.$$

Quae si attentius consideremus, principium, cui demonstratio innititur, huc redire comperimus, ut, si quantitas variabilis primo per differentiationem, deinde vero per variationem proferatur, idem proveniat, ac si ordine inverso primo per variationem, tum vero per differentiationem promoveretur. Veluti in figura ex puncto Y primo per differentiationem pervenitur in Y', hinc vero per variationem in y'; inverso autem ordine primum ex puncto Y per variationem pervenitur in y, hinc vero per differentiationem in punctum y', idem quod ante.

SCHOLION 2

43. Theorema hoc latissime patet; neque enim ad casum duarum variabilium tantum restringitur, sed veritati est etiam consentaneum, quotcunque variabiles in calculum ingrediantur, quandoquidem in demonstratione solius illius variabilis, cuius tam differentiale quam variatio consideratur, ratio habetur sine ullo respectu ad reliquas variabiles. Ne autem hic ulli dubio

locus relinquatur, consideremus superficiem quamcunque, cuius punctum quodvis Z (Fig. 4) per coordinatas ternas $AX = x$, $XY = y$ et $YZ = z$ definiatur; a quo si ad aliud punctum proximum Z' in eadem superficie progrediamur, hae coordinatae suis differentialibus increscent. Tum vero aliam quamcunque superficiem concipiamus proximam, cuius puncta z et z' cum illis Z et Z' conferantur, quod fit per variationem. His positis perspicuum est duplici modo ad punctum z' perveniri posse, altero per variationem ex puncto Z', altero per differentiale ex puncto z, sicque fore

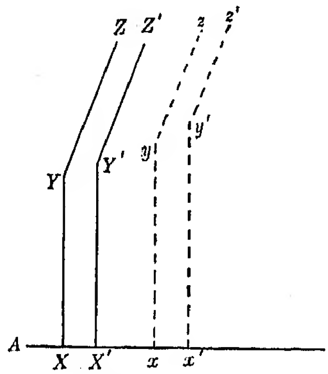

Fig. 4.

$$Ax' = AX' + \delta.\,AX' = Ax + d.\,Ax,$$

$$x'y' = X'Y' + \delta.\,X'Y' = xy + d.\,xy,$$

$$y'z' = Y'Z' + \delta.\,Y'Z' = yz + d.\,yz,$$

quod etiam de omnibus aliis quantitatibus variabilibus ad haec puncta referendis valet. Hinc autem luculenter sequitur fore

$$\delta dx = d\delta x, \quad \delta dy = d\delta y, \quad \delta dz = d\delta z.$$

SCHOLION 3

44. Memorabile prorsus est, quod casu differentialium altioris ordinis signum variationis δ pro lubitu inter signa differentiationis d inscribi possit, atque hinc intelligere licet hanc permutabilitatem locum quoque esse habituram, etiamsi signum variationis δ perinde ac differentiationis d aliquoties repetatur, quod fortasse in aliis speculationibus usu venire posset. Verum in praesenti instituto repetitio variationis δ nullo modo locum habere potest, quoniam lineam vel superficiem tantum cum unica alia sibi proxima comparamus; etsi enim haec generalissime consideratur, ut omnes possibiles itidem proximas in se complectatur, tamen tanquam unica spectatur neque, postquam e principali in proximam transiliverimus, novus transitus in aliam conceditur. Hinc ergo eiusmodi speculationes, quibus variationum variationes essent quaerendae, omnino excluduntur. Vicissim autem variationum differentialia cuiusque ordinis hic admitti debent, et cum in formulis differentialibus, quae quidem significatum habent finitum, ratio differentialium tantum specte-

tur, quae, si binae variabiles sint x et y, huiusmodi positionibus

$$dy = pdx, \quad dp = qdx, \quad dq = rdx \quad \text{etc.}$$

ad formas finitas revocari solent, harum quantitatum p, q, r etc. variationes potissimum assignari necesse est.

PROBLEMA 1

45. *Datis binarum variabilium x et y variationibus δx et δy formulae differentialis $p = \frac{dy}{dx}$ variationem definire.*

SOLUTIO

Cum sit $\delta dy = d\delta y$ et $\delta dx = d\delta x$, variatio quaesita δp per notas differentiationis regulas reperitur, dummodo loco signi differentiationis d scribatur signum variationis δ; unde cum oriatur

$$\delta p = \frac{dx\delta dy - dy\delta dx}{dx^2},$$

erit per conversionem ante demonstratam

$$\delta p = \frac{dxd\delta y - dyd\delta x}{dx^2};$$

ubi cum δx et δy sint variationes ipsarum x et y hincque $\delta x + d\delta x$ et $\delta y + d\delta y$ variationes ipsarum $x + dx$ et $y + dy$, notandum est fore, uti iam observavimus [§ 37],

$$d\delta x = \delta(x + dx) - \delta x \quad \text{et} \quad d\delta y = \delta(y + dy) - \delta y.$$

Idem invenitur ex primis principiis; cum enim valor ipsius p variatus sit $p + \delta p$ isque prodeat, si loco x et y earum valores variati, qui sunt $x + \delta x$ et $y + \delta y$, substituantur, erit

$$p + \delta p = \frac{d(y + \delta y)}{d(x + \delta x)} = \frac{dy + d\delta y}{dx + d\delta x},$$

unde ob $p = \frac{dy}{dx}$ fit

$$\delta p = \delta . \frac{dy}{dx} = \frac{dy + d\delta y}{dx + d\delta x} - \frac{dy}{dx} = \frac{dxd\delta y - dyd\delta x}{dx^2},$$

quoniam in denominatore particula $dxd\delta x$ prae dx^2 evanescit.

COROLLARIUM 1

46. Si, dum per differentialia progredimur, variabiles x et y continuo auctas designemus per x', x'', x''' etc., y', y'', y''' etc., ut sit

$$x' = x + dx \quad \text{et} \quad y' = y + dy,$$

erit

$$d\delta x = \delta x' - \delta x \quad \text{et} \quad d\delta y = \delta y' - \delta y$$

hincque

$$\delta p = \delta . \frac{dy}{dx} = \frac{dx(\delta y' - \delta y) - dy(\delta x' - \delta x)}{dx^2}.$$

COROLLARIUM 2

47. Quoniam variationes ambarum variabilium x et y neutiquam a se invicem pendent, sed prorsus arbitrio nostro relinquuntur, si ipsi x nullas tribuamus variationes, ut sit

$$\delta x = 0 \quad \text{et} \quad \delta x' = 0,$$

erit

$$\delta p = \frac{d\delta y}{dx} = \frac{\delta y' - \delta y}{dx}.$$

COROLLARIUM 3

48. Si praeterea unicae variabili y variationem δy tribuamus, ut sit $\delta y' = 0$, erit $\delta p = -\frac{\delta y}{dx}$, quae hypothesis minime naturae refragatur, quia curvam proximam ita cum principali congruentem assumi licet, ut in unico tantum puncto ab ea discrepet.

SCHOLION

49. Vulgo in solutione problematum isoperimetricorum aliorumque ad id genus pertinentium curva variata ita congruens statui solet, ut tantum in uno quasi elemento discrepet.[1]) Ita si quaerenda sit curva EF (Fig. 5) certa quadam maximi minimive proprietate gaudens, unicum punctum Y in locum proximum y transferri solet, ut curva variata $EMyY'F$ tantum in intervallo minimo MY' a quaesita deflectat, ita ut positis $AX = x$ et $XY = y$ sit pro variata curva

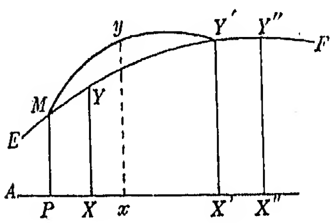

Fig. 5.

1) Cf. notam p. 393. F. E.

$$Ax = x + \delta x \quad \text{et} \quad xy - y + \delta y$$

seu

$$\delta x = Ax - AX \quad \text{et} \quad \delta y = xy - XY,$$

pro sequentibus vero punctis, ad quae differentialia ducunt, sit ubique

$$\delta x' = 0, \quad \delta y' = 0, \quad \delta x'' = 0, \quad \delta y'' = 0 \quad \text{etc.}$$

itemque pro antecedentibus. Quin etiam ad calculi commodum variatio $Xx = \delta x$ nulla sumi solet, ut omnis variatio ad solum elementum δy perducatur, quo casu utique habebitur $\delta p = -\frac{\delta y}{dx}$, haecque unica variatio utique sufficit ad problemata huius generis, quae quidem fuerint tractata, resolvenda.

Verum si, uti hic instituimus, haec problemata latius extendimus, ut curva quaesita circa initium et finem certas determinationes recipere queat, utique necessarium est calculum variationum quam generalissime absolvere atque in omnibus curvae punctis variationes indefinitas coordinatis tribuere. Quod etiam maxime est necessarium, si huiusmodi investigationes ad lineas curvas non continuas accommodare velimus.

PROBLEMA 2

50. *Datis binarum variabilium x et y variationibus δx et δy, si ponatur $dy = p\,dx$ et $dp = q\,dx$, invenire variationem quantitatis q seu valorem ipsius δq.*

SOLUTIO

Cum sit $q = \frac{dp}{dx}$, erit pro valore variato

$$q + \delta q = \frac{d(p + \delta p)}{d(x + \delta x)} = \frac{dp + d\delta p}{dx + d\delta x},$$

unde auferendo quantitatem q relinquitur

$$\delta q = \frac{dx\,d\delta p - dp\,d\delta x}{dx^2},$$

quae variatio ergo etiam ex differentiatione formulae $q = \frac{dp}{dx}$ resultat, si more consueto differentiatio instituatur, loco vero signi differentialis d scribatur signum variationis δ; ubi quidem meminisse iuvabit esse

$$\delta dx = d\delta x \quad \text{et} \quad \delta dp = d\delta p.$$

Supra autem invenimus ob $p=\frac{dy}{dx}$ esse

$$\delta p=\frac{dx\,d\delta y-dy\,d\delta x}{dx^2},$$

unde porro per consuetam differentiationem valor ipsius $d\delta p$, scilicet differentiale ipsius δp colligitur.

COROLLARIUM 1

51. Cum sit $\frac{dy}{dx}=p$ et $\frac{dp}{dx}=q$, erit primo

$$\delta p=\frac{d\delta y}{dx}-\frac{p\,d\delta x}{dx},$$

tum vero

$$\delta q=\frac{d\delta p}{dx}-\frac{q\,d\delta x}{dx}.$$

Pro usu autem futuro praestat hic particulam $d\delta p$ relinqui quam eius valorem ex praecedente formula erui.

COROLLARIUM 2

52. Interim tamen cum prior per differentiationem det

$$d\delta p=\frac{dd\delta y}{dx}-\frac{ddx\,d\delta y}{dx^2}-\frac{p\,dd\delta x}{dx}-q\,d\delta x+\frac{p\,ddx\,d\delta x}{dx^2},$$

hoc valore substituto prodit

$$\delta q=\frac{dd\delta y}{dx^2}-\frac{ddx\,d\delta y}{dx^3}-\frac{p\,dd\delta x}{dx^2}-\frac{2q\,d\delta x}{dx}+\frac{p\,ddx\,d\delta x}{dx^3}.$$

COROLLARIUM 3

53. Quodsi soli variabili y variationes tribuantur, ut particulae δx et quae inde derivantur evanescant, habebimus

$$\delta p=\frac{d\delta y}{dx}\quad\text{et}\quad\delta q=\frac{d\delta p}{dx}=\frac{dd\delta y}{dx^2}-\frac{ddx\,d\delta y}{dx^3},$$

ac si differentiale dx constans accipiatur, erit $\delta q=\frac{dd\delta y}{dx^2}$.

SCHOLION 1

54. Quo haec facilius intelligantur, consideremus in curva EF (Fig. 5, p. 389) per relationem inter variabiles $AX = x$ et $XY = y$ plura puncta Y, Y', Y'' etc. secundum differentialia continuo promota, ut sit

$$AX = x, \quad AX' = x + dx, \quad AX'' = x + 2dx + ddx,$$
$$AX''' = x + 3dx + 3ddx + d^3x,$$
$$XY = y, \quad X'Y' = y + dy, \quad X''Y'' = y + 2dy + ddy,$$
$$X'''Y''' = y + 3dy + 3ddy + d^3y,$$

quae differentialia cuiusque ordinis indicantes ita brevitatis gratia repraesententur

$$AX = x, \quad AX' = x', \quad AX'' = x'', \quad AX''' = x''' \quad \text{etc.},$$
$$XY = y, \quad X'Y' = y', \quad X''Y'' = y'', \quad X'''Y''' = y''' \quad \text{etc.},$$

quibus singulis suae variationes nullo modo a se invicem pendentes tribui concipiantur, ita ut omnes istae variationes

$$\delta x, \quad \delta x', \quad \delta x'', \quad \delta x''' \quad \text{etc.},$$
$$\delta y, \quad \delta y', \quad \delta y'', \quad \delta y''' \quad \text{etc.}$$

a lubitu nostro pendentes tanquam cognitae spectari queant. His iam ita constitutis differentialia cuiusque ordinis variationum in hunc modum repraesentabuntur, ut sit

$$d\delta x = \delta x' - \delta x, \quad dd\delta x = \delta x'' - 2\delta x' + \delta x, \quad d^3\delta x = \delta x''' - 3\delta x'' + 3\delta x' - \delta x,$$
$$d\delta y = \delta y' - \delta y, \quad dd\delta y = \delta y'' - 2\delta y' + \delta y, \quad d^3\delta y = \delta y''' - 3\delta y'' + 3\delta y' - \delta y.$$

Quodsi iam unicum punctum curvae Y variari sumamus, erit

$$d\delta x = -\delta x, \quad dd\delta x = +\delta x, \quad d^3\delta x = -\delta x \quad \text{etc.},$$
$$d\delta y = -\delta y, \quad dd\delta y = +\delta y, \quad d^3\delta y = -\delta y \quad \text{etc.}$$

hincque

$$\delta p = -\frac{\delta y}{dx} + \frac{p\,\delta x}{dx}$$

et

$$\delta q = \frac{\delta y}{dx^2} + \frac{ddx\,\delta y}{dx^3} - \frac{p\,\delta x}{dx^2} + \frac{2q\,\delta x}{dx} - \frac{p\,ddx\,\delta x}{dx^3},$$

ubi omissis partibus reliquarum respectu evanescentibus erit

$$\delta q = \delta y \cdot \frac{1}{dx^2} - \delta x \cdot \frac{p}{dx^2}.$$

Denique si soli applicatae $XY = y$ variatio tribuatur, habebitur[1])

$$\delta p = -\frac{1}{dx}\delta y \quad \text{et} \quad \delta q = \frac{1}{dx^2}\delta y.$$

SCHOLION 2

55. Hinc patet, si in unico curvae puncto variatio statuatur, insigniter contra recepta differentialium principia impingi, cum variationum differentialia superiora neutiquam prae inferioribus evanescant, sed iugiter eundem valorem retineant atque adeo variationes quantitatum p et q in infinitum excrescant, siquidem infinite parva δx et δy ex eodem ordine quo differentialia dx et dy assumantur. Quin etiam hinc in calculo maxime cavendum est, ne in enormes errores praecipitemur, cum calculi praecepta legi continuitatis innitantur, qua lineae curvae continuo puncti fluxu describi concipiuntur, ita ut in earum curvatura nusquam saltus agnoscatur. Quodsi autem unicum curvae punctum Y (Fig. 5, p. 389) in y diducatur reliquo curvae tractu praeter bina quasi elementa My et yY' invariato relicto, evidens est curvaturae ingentem irregularitatem induci, cum vulgares calculi regulae non amplius applicari queant. Cui incommodo ut occurramus, tutissimum erit remedium, ut singulis curvae punctis mente saltem variationes tribuantur, quae continuitatis quapiam lege contineantur, neque ante irregularitas in calculo admittatur, quam omnes differentiationes et integrationes fuerint peractae, hocque modo saltem species continuitatis in calculo retineatur. Quamvis ergo variationum differentialia

$$d\delta y, \quad dd\delta y, \quad d^3\delta y \quad \text{etc.},$$

item

$$d\delta x, \quad dd\delta x, \quad d^3\delta x \quad \text{etc.}$$

forte in facta hypothesi ad simplices variationes revocare liceat, tamen expedit illas formas in calculo retineri ad easque sequentes integrationes accommodari; atque huc etiam redeunt operationes, quas olim[1]), cum idem argumentum de inveniendis curvis maximi minimive proprietate praeditis tractassem, expedire docueram.

1) *Methodus inveniendi lineas curvas*, cap. II § 1, 21, 56; vide notam p. 375. F. E.

PROBLEMA 3

56. *Datis binarum variabilium x et y variationibus δx et δy rationum inter differentialia cuiuscunque gradus variationes investigare.*

SOLUTIO

Quaestio huc redit, ut positis continuo

$$dy = pdx, \quad dp = qdx, \quad dq = rdx, \; dr = sdx \quad \text{etc.}$$

quantitatum p, q, r, s etc. variationes assignentur, cum ad has quantitates omnes differentialium cuiuscunque ordinis rationes, quae quidem finitis valoribus continentur, reducantur. Ac de harum quidem duabus primis p et q iam vidimus esse

$$\delta p = \frac{d\delta y}{dx} - \frac{pd\delta x}{dx} \quad \text{et} \quad \delta q = \frac{d\delta p}{dx} - \frac{qd\delta x}{dx}.$$

Quoniam igitur porro est

$$r = \frac{dq}{dx} \quad \text{et} \quad s = \frac{dr}{dx} \quad \text{etc.},$$

harum variationes simili modo per differentiationis regulas inveniuntur

$$\delta r = \frac{d\delta q}{dx} - \frac{rd\delta x}{dx}, \quad \delta s = \frac{d\delta r}{dx} - \frac{sd\delta x}{dx} \quad \text{etc.},$$

ubi, si lubuerit, loco $d\delta p$, $d\delta q$, $d\delta r$ etc. differentialia variationum δp, δq, δr etc. ante inventarum substitui possunt. Hoc autem non solum in formulas nimis prolixas induceret, sed etiam, uti ex sequentibus patebit, ne quidem est necessarium, cum hinc multo facilius omnes reductiones, quibus opus erit, institui queant.

COROLLARIUM 1

57. Si soli variabili y variationes tribuantur seu manentibus abscissis x tantum applicatae y suis variationibus augeantur, habebimus

$$\delta p = \frac{d\delta y}{dx}, \quad \delta q = \frac{d\delta p}{dx}, \quad \delta r = \frac{d\delta q}{dx}, \quad \delta s = \frac{d\delta r}{dx} \quad \text{etc.}$$

COROLLARIUM 2

58. Quodsi praeterea omnia ipsius x incrementa dx aequalia capiantur seu elementum dx constans statuatur, substitutis differentialibus praecedentium formularum in sequentibus obtinebitur

$$\delta p = \frac{d\delta y}{dx}, \quad \delta q = \frac{dd\delta y}{dx^2}, \quad \delta r = \frac{d^3\delta y}{dx^3}, \quad \delta s = \frac{d^4\delta y}{dx^4} \quad \text{etc.}$$

COROLLARIUM 3

59. Si solis abscissis x variationes tribuantur, ut variatio δy cum omnibus derivatis evanescat, simulque elementum dx constans capiatur, singulae hae variationes ita se habebunt

$$\delta p = \frac{-pd\delta x}{dx}, \quad \delta q = \frac{-pdd\delta x}{dx^2} - \frac{2qd\delta x}{dx},$$

$$\delta r = \frac{-pd^3\delta x}{dx^3} - \frac{3qdd\delta x}{dx^2} - \frac{3rd\delta x}{dx},$$

$$\delta s = \frac{-pd^4\delta x}{dx^4} - \frac{4qd^3\delta x}{dx^3} - \frac{6rdd\delta x}{dx^2} - \frac{4sd\delta x}{dx}$$

etc.

COROLLARIUM 4

60. Etiamsi ergo hoc casu elementum dx constans accipiatur, tamen hic occurrunt differentialia cuiusque ordinis variationis δx; cuius rei ratio est, quod variationes valorum ipsius x continuo ulterius promotorum x', x'' etc. neutiquam a differentialibus pendere statuuntur.

SCHOLION

61. Quando autem placuerit soli variabili x variationes tribuere, tum omnino praestat variabiles x et y inter se permutari atque huiusmodi potius positionibus uti

$$dx = pdy, \quad dp = qdy, \quad dq = rdy \quad \text{etc.},$$

quibus species differentialium tollatur; tum vero sumto elemento dy constante similes formulae simpliciores pro variationibus quantitatum p, q, r etc. re-

periuntur atque Corollario 2. Ceterum quo calculus ad omnes casus accommodari queat, semper expedit utrique variabili suas variationes tribui; etsi enim tum formae multo perplexiores prodeant, praecipue si evolvantur, tamen calculum prosequendo tam egregia se offerunt compendia, ut in fine calculus vix fiat operosior neque huius prolixitatis taedeat. Ad problemata ergo magis generalia ad hoc caput pertinentia progrediamur.

PROBLEMA 4

62. *Datis duarum variabilium x et y variationibus δx et δy formulae cuiuscunque finitae V tam ex illis variabilibus ipsis quam earum differentialibus cuiuscunque ordinis conflatae variationem invenire.*

SOLUTIO

Cum V sit quantitas valorem habens finitum, ponendo

$$dy = p\,dx, \quad dp = q\,dx, \quad dq = r\,dx, \quad dr = s\,dx \quad \text{etc.}$$

differentialia inde tollentur prodibitque pro V functio ex quantitatibus finitis formata x, y, p, q, r, s etc. Quaecunque ergo sit ratio compositionis, eius differentiale semper huiusmodi habebit formam

$$dV = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + S\,ds + \text{etc.}$$

horum membrorum numero existente eo maiore, quo altiora differentialia ingrediuntur in V. Quodsi vero huius formulae V variatio δV fuerit indaganda, ea obtinetur, si loco quantitatum variabilium x, y, p, q, r etc. eaedem suis variationibus auctae substituantur et a forma resultante ipsa quantitas V auferatur, ex quo intelligitur variationem ope consuetae differentiationis inveniri signo tantum differentialis d in signum variationis δ mutato. Quare cum differentiale supra iam sit exhibitum, impetrabimus variationem quaesitam

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.};$$

quemadmodum autem variationes δp, δq, δr, δs etc. per variationes sumtas δx et δy determinentur, iam supra [§ 56] est ostensum.

COROLLARIUM 1

63. Si hic substituamus valores ante inventos, obtinebimus variationem quaesitam ita expressam

$$\delta V = M\delta x + N\delta y + \frac{1}{dx}(Pd\delta y + Qd\delta p + Rd\delta q + Sd\delta r + \text{etc.})$$
$$- \frac{d\delta x}{dx}(Pp + Qq + Rr + Ss + \text{etc.}).$$

COROLLARIUM 2

64. Si variabili x nulla plane tribuatur variatio atque insuper elementum dx constans accipiatur, tum quantitatis propositae V variatio ita prodibit expressa

$$\delta V = N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} + \text{etc.}$$

SCHOLION

65. In his formis saltem species homogeneitatis in differentialibus spectatur, siquidem δx et δy ad ordinem differentialium referantur; quod longe secus eveniret, si eo casu, quo unicum curvae punctum variatur, statim vellemus loco differentialium variationum valores supra (§ 54) exhibitos substituere, quo quippe pacto idea integrationis, qua hae formulae deinceps indigent, excluderetur. Ceterum patet, quomodo inventio variationum ad consuetam differentiationem revocetur, dum totum discrimen in hoc tantum est situm, ut loco variationum δp, δq, δr etc. valores iam ante assignati, quos quidem ipsos quoque per consuetam differentiationem elicuimus, substituantur. Conveniet autem hanc operationem aliquot exemplis illustrari, quo clarius indoles totius huius tractationis percipiatur.

EXEMPLUM 1

66. *Formulae subtangentem exprimentis* $\frac{ydx}{dy}$ *variationem invenire.*

Ob $dy = pdx$ haec formula fit $\frac{y}{p}$, unde eius variatio $\frac{\delta y}{p} - \frac{y\delta p}{pp}$, ubi loco δp valore substituto fit ea

$$\frac{\delta y}{p}-\frac{yd\delta y}{pp\,dx}+\frac{yd\delta x}{pdx}=\frac{dx}{dy}\delta y-\frac{ydx}{dy^2}d\delta y+\frac{y}{dy}d\delta x,$$

quae postrema forma immediate ex differentiatione formulae propositae nascitur.

EXEMPLUM 2

67. *Formulae ipsam tangentem exprimentis* $\frac{y\sqrt{(dx^2+dy^2)}}{dy}$ *variationem invenire.*

Positio $dy=pdx$ praebet hanc formam finitam $\frac{y}{p}\sqrt{(1+pp)}$, unde variatio quaesita est

$$\frac{\delta y}{p}\sqrt{(1+pp)}-\frac{y\delta p}{pp\sqrt{(1+pp)}},$$

quae transformatur in hanc

$$\frac{\sqrt{(dx^2+dy^2)}}{dy}\delta y-\frac{ydx}{dy^2\sqrt{(dx^2+dy^2)}}(dxd\delta y-dyd\delta x).$$

EXEMPLUM 3

68. *Formulae radium curvedinis exprimentis* $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy}$ *variationem definire.*

Posito $dy=pdx$ et $dp=qdx$ haec formula transit in hanc $\frac{(1+pp)^{\frac{3}{2}}}{q}$, cuius propterea variatio est

$$\frac{3p\delta p}{q}\sqrt{(1+pp)}-\frac{\delta q}{qq}(1+pp)^{\frac{3}{2}},$$

ubi quidem substitutioni valorum ante inventorum non immoror.

PROBLEMA 5

69. *Datis duarum quantitatum variabilium x et y variationibus δx et δy formulae tam ex illis variabilibus quam earum differentialibus cuiuscunque ordinis conflatae, sive fuerit infinita sive infinite parva, variationem investigare.*

SOLUTIO

Positis ut hactenus $dy=pdx$, $dp=qdx$, $dq=rdx$ etc. formula semper reducetur ad huiusmodi formam Vdx^n, ubi V sit functio finita quantitatum

x, y, p, q, r etc., exponens vero n sive positivus sive negativus, ita ut priori casu formula sit infinite parva, posteriori vero infinite magna. Ponamus igitur differentiationem ordinariam dare

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

unde simul eius variatio habetur. Cum igitur formae propositae variatio sit

$$nVdx^{n-1}d\delta x + dx^n \delta V,$$

erit utique haec variatio, quam quaerimus,

$$nVdx^{n-1}d\delta x + dx^n(M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}),$$

ubi ex superioribus [§ 56] hos valores substitui oportet

$$\delta p = \frac{d\delta y - pd\delta x}{dx}, \quad \delta q = \frac{d\delta p - qd\delta x}{dx},$$

$$\delta r = \frac{d\delta q - rd\delta x}{dx}, \quad \delta s = \frac{d\delta r - sd\delta x}{dx}$$

etc.

Quae cum per se sint perspicua, nulla ampliori explicatione indigent simulque hoc caput penitus absolutum videtur.

CAPUT III

DE VARIATIONE FORMULARUM INTEGRALIUM SIMPLICIUM DUAS VARIABILES INVOLVENTIUM

DEFINITIO 6

70. *Formulam integralem simplicem hic appello, quae nulla alia integralia in se involvit, sed simpliciter integrale refert formulae differentialis praeter binas variabiles quaecunque earum differentialia complectentis.*

COROLLARIUM 1

71. Si ergo x et y sint binae variabiles, formula integralis $\int W$ erit simplex, si expressio W praeter has variabiles tantum earum differentialia, cuiuscunque fuerint ordinis, contineat neque praeterea alias formulas integrales in se implicet.

COROLLARIUM 2

72. Quodsi ergo statuamus

$$dy = pdx, \quad dp = qdx, \quad dq = rdx \quad \text{etc.},$$

ut species differentialium tollatur, quoniam integratio requirit formulam differentialem, expressio illa W semper reducetur ad huiusmodi formam Vdx existente V functione quantitatum x, y, p, q etc.

COROLLARIUM 3

73. Cum igitur formula integralis simplex sit huiusmodi $\int Vdx$, ubi V est functio quantitatum x, y, p, q, r etc., eius indolem commodissime eius differentiale repraesentabit, si dicamus esse

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

SCHOLION

74. Distinguo hic formulas integrales simplices a complicatis, in quibus integratio proponitur eiusmodi formularum differentialium, quae iam ipsae unam pluresve formulas integrales involvunt. Veluti si littera s denotet integrale

$$\int \sqrt{(dx^2 + dy^2)} = \int dx \sqrt{(1 + pp)}$$

atque quantitas V praeter illas quantitates etiam hanc s contineat, formula integralis $\int V dx$ merito censetur complicata; cuius variatio singularia praecepta postulat deinceps [cap. IV] exponenda. Hoc autem capite primo methodum formularum integralium simplicium variationes inveniendi tradere constitui.

THEOREMA 2

75. *Variatio formulae integralis $\int W$ semper aequalis est integrali variationis eiusdem formulae differentialis, cuius integrale proponitur, seu est*

$$\delta \int W = \int \delta W.$$

DEMONSTRATIO

Cum variatio sit excessus, quo valor variatus cuiusque quantitatis superat eius valorem naturalem, perpendamus formulae propositae $\int W$ valorem variatum, quem obtinet, si loco variabilium x et y earundem valores suis variationibus δx et δy aucti substituantur. Cum autem tum quantitas W abeat in $W + \delta W$, formae propositae valor variatus erit

$$\int (W + \delta W) = \int W + \int \delta W;$$

unde cum sit

$$\delta \int W = \int (W + \delta W) - \int W,$$

habebimus

$$\delta \int W = \int \delta W,$$

unde patet variationem integralis aequari integrali variationis.

Idem etiam hoc modo ostendi potest. Ponatur $\int W = w$, ita ut quaerenda sit variatio δw. Quia ergo sumtis differentialibus est $dw = W$, capiantur nunc variationes eritque

$$\delta dw = \delta W = d\delta w$$

ob $\delta dw = d\delta w$. Nunc vero aequatio $d\delta w = \delta W$ denuo integrata praebet

$$\delta w = \int \delta W = \delta \int W.$$

COROLLARIUM 1

76. Proposita ergo hac formula integrali $\int V dx$ eius variatio $\delta \int V dx$ erit

$$= \int \delta (V dx) = \int (V \delta dx + dx \delta V)$$

hincque ob $\delta dx = d\delta x$ habebitur

$$\delta \int V dx = \int V d\delta x + \int dx \delta V.$$

COROLLARIUM 2

77. Posito $\delta x = \omega$, ut sit $d\delta x = d\omega$, quia est

$$\int V d\omega = V\omega - \int \omega dV,$$

in priori membro differentiale variationis dx exuitur fietque

$$\delta \int V dx = V\delta x - \int dV \delta x + \int dx \delta V,$$

ubi prima pars ab integratione est immunis.

SCHOLION

78. Quemadmodum supra [§ 37] ostendimus signa differentiationis d cum signo variationis δ expressioni cuicunque praefixa inter se pro lubitu permutari posse, ita nunc videmus signum integrationis $\int$ cum signo variationis δ permutari posse, cum sit

$$\delta \int W = \int \delta W.$$

Atque hoc etiam ad integrationes repetitas patet, ut, si proposita fuerit talis formula $\int\int W$, eius variatio his modis exhiberi possit

$$\delta\int\int W = \int\delta\int W = \int\int\delta W$$

ideoque variatio formularum integralium ad variationes expressionum nullam amplius integrationem involventium reducatur, pro quibus inveniendis iam supra praecepta sunt tradita.

PROBLEMA 6

79. *Propositis binarum variabilium x et y variationibus δx et δy, si positis*

$$dy = p\,dx, \quad dp = q\,dx, \quad dq = r\,dx \quad etc.$$

fuerit V functio quaecunque quantitatum x, y, p, q, r etc., formulae integralis $\int V dx$ variationem investigare.

SOLUTIO

Modo vidimus (§ 77) huius formulae integralis variationem ita exprimi, ut sit

$$\delta\int V dx = V\delta x - \int dV\delta x + \int dx\delta V.$$

Iam ad variationem δV elidendam, cum sit V functio quantitatum x, y, p, q, r etc., statuamus eius differentiale esse

$$dV = M dx + N dy + P dp + Q dq + R dr + \text{etc.}$$

ac simili modo eius variatio ita erit expressa

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.},$$

quibus valoribus substitutis consequimur variationem quaesitam

$$\delta\int V dx = V\delta x + \int dx(M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.})$$
$$- \int\delta x(M dx + N dy + P dp + Q dq + R dr + \text{etc.});$$

ubi cum partes ab M pendentes se destruant, erit partibus secundum litteras

N, P, Q, R etc. separatis variatio

$$\begin{aligned}\delta\int Vdx = V\delta x &+ \int N(dx\delta y - dy\delta x) + \int P(dx\delta p - dp\delta x)\\ &+ \int Q(dx\delta q - dq\delta x) + \int R(dx\delta r - dr\delta x)\\ &+ \text{etc.},\end{aligned}$$

ubi est, uti supra [§ 56] invenimus,

$$dx\delta p = d\delta y - pd\delta x, \quad dx\delta q = d\delta p - qd\delta x, \quad dx\delta r = d\delta q - rd\delta x \quad \text{etc.},$$

quibus valoribus substitutis ob $dy = pdx$ obtinetur

$$\begin{aligned}\delta\int Vdx = V\delta x &+ \int Ndx(\delta y - p\delta x) + \int Pd.(\delta y - p\delta x)\\ &+ \int Qd.(\delta p - q\delta x) + \int Rd.(\delta q - r\delta x)\\ &+ \text{etc.}\end{aligned}$$

Ad hanc expressionem ulterius reducendam notetur esse

$$\delta p - q\delta x = \frac{d\delta y - pd\delta x - dp\delta x}{dx} = \frac{d.(\delta y - p\delta x)}{dx},$$

$$\delta q - r\delta x = \frac{d\delta p - qd\delta x - dq\delta x}{dx} = \frac{d.(\delta p - q\delta x)}{dx},$$

$$\delta r - s\delta x = \frac{d\delta q - rd\delta x - dr\delta x}{dx} = \frac{d.(\delta q - r\delta x)}{dx}$$

etc.,

quo pacto quaevis formula ad praecedentem reducitur; unde, si brevitatis gratia ponamus $\delta y - p\delta x = \omega$, erit, ut sequitur,

$$\delta y - p\delta x = \omega,$$

$$\delta p - q\delta x = \frac{d\omega}{dx},$$

$$\delta q - r\delta x = \frac{1}{dx}d.\frac{d\omega}{dx},$$

$$\delta r - s\delta x = \frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx}$$

etc.

sicque variationibus litterarum derivatarum p, q, r etc. ex calculo exclusis adipiscimur variationem quaesitam

$$\delta\int Vdx = V\delta x + \int Ndx\omega + \int Pd\omega + \int Qd.\frac{d\omega}{dx} + \int Rd.\frac{1}{dx}d.\frac{d\omega}{dx}$$
$$+ \int Sd.\frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx} + \int Td.\frac{1}{dx}d.\frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx} + \text{etc.},$$

cuius formae lex progressionis est manifesta, cuiuscunque gradus differentialia in formulam V ingrediantur.

COROLLARIUM 1

80. Huius igitur variationis pars prima $V\delta x$ a signo integrationis est immunis atque adeo solam variationem δx involvit, reliquae vero partes utramque perpetuo eodem modo iunctam et in littera $\omega = \delta y - p\delta x$ comprehensam continent.

COROLLARIUM 2

81. Secunda pars

$$\int Ndx \cdot \omega = \int N\omega dx$$

commodius exprimi nequit, tertia vero $\int Pd\omega$ commodius ita exprimi videtur, ut sit

$$\int Pd\omega = P\omega - \int \omega dP$$

ac post signum integrale iam ipsa quantitas ω reperiatur.

COROLLARIUM 3

82. Quarta pars $\int Qd.\frac{d\omega}{dx}$ simili modo reducitur ad

$$Q\frac{d\omega}{dx} - \int dQ\frac{d\omega}{dx}$$

hocque membrum posterius, cum sit $\int \frac{dQ}{dx}d\omega$, porro praebet

$$\frac{dQ}{dx}\omega - \int \omega d.\frac{dQ}{dx},$$

ita ut quarta pars resolvatur in haec membra

$$Q\frac{d\omega}{dx}-\frac{dQ}{dx}\omega+\int\omega d.\frac{dQ}{dx}.$$

COROLLARIUM 4

83. Quinta pars

$$\int Rd.\frac{1}{dx}d.\frac{d\omega}{dx}$$

reducitur primo ad

$$R\frac{1}{dx}d.\frac{d\omega}{dx}-\int\frac{dR}{dx}d.\frac{d\omega}{dx},$$

tum vero posterius membrum ad

$$\frac{dR}{dx}\frac{d\omega}{dx}-\int\frac{1}{dx}d.\frac{dR}{dx}\cdot d\omega$$

hocque tandem ulterius ad

$$\frac{1}{dx}d.\frac{dR}{dx}\cdot\omega-\int\omega d.\frac{1}{dx}d.\frac{dR}{dx},$$

ita ut haec quinta pars iam ita exprimatur

$$R\frac{1}{dx}d.\frac{d\omega}{dx}-\frac{dR}{dx}\cdot\frac{d\omega}{dx}+\frac{1}{dx}d.\frac{dR}{dx}\cdot\omega-\int\omega d.\frac{1}{dx}d.\frac{dR}{dx}.$$

COROLLARIUM 5

84. Simili modo sexta pars

$$\int Sd.\frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx}$$

ita reperitur expressa

$$S\frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx}-\frac{dS}{dx}\cdot\frac{1}{dx}d.\frac{d\omega}{dx}+\frac{1}{dx}d.\frac{dS}{dx}\cdot\frac{d\omega}{dx}-\frac{1}{dx}d.\frac{1}{dx}d.\frac{dS}{dx}\cdot\omega$$
$$+\int\omega d.\frac{1}{dx}d.\frac{1}{dx}d.\frac{dS}{dx}.$$

PROBLEMA 7

85. *Positis* $dy = pdx$, $dp = qdx$, $dq = rdx$, $dr = sdx$ *etc. si* V *fuerit functio quaecunque quantitatum* x, y, p, q, r, s *etc., ita ut sit*

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.},$$

formulae integralis $\int Vdx$ *variationem ex utriusque variabilis* x *et* y *variatione natam ita exprimere, ut post signum integrale nulla occurrant variationum differentialia.*

SOLUTIO

In corollariis praecedentis problematis iam omnia ita sunt ad hunc scopum praeparata, ut nihil aliud opus sit, nisi ut transformationes singularum partium in ordinem redigantur, quo pacto duplicis generis partes obtinentur, uno continente formulas integrales, quas quidem omnes in eandem summam colligere licet, altero partes absolutas, quas ita in membra distribuemus, ut secundum ipsas variationes δx et δy earumque differentialia cuiusque gradus procedant. Posita autem brevitatis gratia formula $\delta y - p\delta x = \omega$ variatio quaesita ita se habebit

$$\begin{aligned}\delta\int Vdx = \int \omega dx\Big(N - \frac{dP}{dx} + \frac{1}{dx}d.\frac{dQ}{dx} - \frac{1}{dx}d.\frac{1}{dx}d.\frac{dR}{dx} + \frac{1}{dx}d.\frac{1}{dx}d.\frac{1}{dx}d.\frac{dS}{dx} - \text{etc.}\Big)\\ + V\delta x + \omega\left(P - \frac{dQ}{dx} + \frac{1}{dx}d.\frac{dR}{dx} - \frac{1}{dx}d.\frac{1}{dx}d.\frac{dS}{dx} + \text{etc.}\right)\\ + \frac{d\omega}{dx}\left(Q - \frac{dR}{dx} + \frac{1}{dx}d.\frac{dS}{dx} - \text{etc.}\right)\\ + \frac{1}{dx}d.\frac{d\omega}{dx}\left(R - \frac{dS}{dx} + \text{etc.}\right)\\ + \frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx}(S - \text{etc.})\\ + \text{etc.},\end{aligned}$$

cuius formae indoles ex sola inspectione statim est manifesta, ut uberiori illustratione non sit opus.

COROLLARIUM 1

86. Haec expressio multo simplicior redditur, si elementum dx capiatur constans, quo quidem amplitudo expressionis nequaquam restringitur; tum

enim fiet

$$\begin{aligned}\delta\int V dx = \int \omega\, dx\left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right)\\ + V\delta x + \omega\left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right)\\ + \frac{d\omega}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right)\\ + \frac{dd\omega}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right)\\ + \frac{d^3\omega}{dx^3}(S - \text{etc.}).\\ + \text{etc.}\end{aligned}$$

COROLLARIUM 2

87. Si quaestio sit de linea curva, prima pars integralis valorem per totam curvam ab initio usque ad terminum, ubi coordinatae x et y subsistunt, congregat simul omnes variationes in singulis curvae punctis factas complectens, dum reliquae partes absolutae tantum per variationes in extremitate curvae factas definiuntur.

COROLLARIUM 3

88. Si curvam coordinatis x et y definitam ut datam spectemus aliaque curva ab ea infinite parum discrepans consideretur, dum in singulis punctis utrique coordinatae variationes quaecunque tribuantur, expressio inventa indicat, quantum formulae integralis $\int V dx$ valor ex curva variata collectus superat eiusdem valorem ex ipsa curva data desumtum.

COROLLARIUM 4

89. Cum sit $\omega = \delta y - p\delta x$, patet hanc quantitatem ω evanescere, si in singulis punctis variationes δx et δy ita accipiantur, ut sit

$$\delta y : \delta x = p : 1 = dy : dx.$$

Hoc igitur casu curva variata plane non discrepat a data ac tota variatio formulae $\int V dx$ reducitur ad $V\delta x$.

SCHOLION 1

90. Variatio haec pro formula integrali $\int V dx$ inventa statim suppeditat regulam, quam olim[1]) tradidi pro curva invenienda, in qua valor eiusdem formulae integralis sit maximus vel minimus. Illa enim regula postulat, ut haec expressio

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}$$

nihilo aequalis statuatur. Hic autem statim evidens est ad id, ut variatio formulae $\int V dx$ evanescat, quemadmodum natura maximorum et minimorum exigit, ante omnia requiri, ut prima pars signo integrali contenta evanescat, ex quo fit

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0.$$

Praeterea vero etiam partes absolutas nihilo aequari oportet, in quo applicatio ad utrumque curvae terminum continetur. Ipsa enim curvae natura per illam aequationem exprimitur; quae cum ob differentialia altioris gradus totidem integrationes totidemque constantes arbitrarias assumat, harum constantium determinationi illae partes absolutae inserviunt, ut tam in initio quam in fine quaesita curva certis conditionibus respondeat, veluti ad datas lineas curvas terminetur. Ac si aequatio illa fuerit differentialis ordinis quarti vel adeo altioris, partium quoque absolutarum numerus augetur, quibus effici potest, ut curva quaesita non solum utrinque ad datas lineas terminetur, sed ibidem quoque certa directio, quin etiam, si ad altiora differentialia assurgat, certa curvaminis lex praescribi queat. Semper autem applicationem faciendo pulcherrime evenire solet, ut ipsa quaestionum indoles eiusmodi conditiones involvat, quibus per partes absolutas commodissime satisfieri possit.

SCHOLION 2

91. Quanta autem mysteria in hac forma, quam pro variatione formulae integralis $\int V dx$ invenimus, lateant, in eius applicatione ad maxima et minima multo luculentius declarare licet; hic tantum observo partem integralem necessario in istam variationem ingredi. Cum enim rem in latissimo sensu

1) *Methodus inveniendi lineas curvas*, cap. II § 56; vide notam p. 375. F. E.

simus complexi, ut in singulis curvae punctis utrique variabili x et y variationes quascunque nulla plane lege inter se connexas tribuerimus, fieri omnino nequit, ut variatio toti curvae conveniens non simul ab omnibus variationibus intermediis pendeat, quippe quibus aliter constitutis necesse est, ut inde totius curvae variatio mutationem perpetiatur. Atque in hoc variatio formularum integralium potissimum dissidet a variatione eiusmodi expressionum, quales in superiori capite consideravimus, quae unice a variatione ultimis elementis tributa pendet. Ex quo luculenter sequitur, si forte quantitas V ita fuerit comparata, ut formula differentialis Vdx integrationem admittat nulla stabilita relatione inter variabiles x et y sicque [formula] integralis $\int Vdx$ sit functio absoluta quantitatum x, y, p, q, r etc., tum etiam eius variationem tantum a variatione extremorum elementorum pendere posse sicque partem variationis integralem plane in nihilum abire debere, ex quo sequens insigne theorema colligitur.

THEOREMA 3

92. *Posito $dy = pdx$, $dp = qdx$, $dq = rdx$, $dr = sdx$ etc. si V fuerit eiusmodi functio ipsarum x, y, p, q, r, s etc., ut posito eius differentiali*

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.}$$

fuerit

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

sumto elemento dx constante, tum formula differentialis Vdx per se erit integrabilis nulla stabilita relatione inter variabiles x et y, ac vicissim.

DEMONSTRATIO

Si fuerit

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0,$$

tum formulae integralis $\int Vdx$ variatio nullam implicat formulam integralem ideoque pro quovis situ coordinatarum x et y a solis variationibus, quae ipsis in extremo termino tribuuntur, pendet, quod fieri neutiquam posset, si formula Vdx integrationem respueret, proptera quod tum variatio insuper ab omnibus variationibus intermediis simul necessario penderet; unde sequitur,

quoties aequatio illa locum habet, toties formulam Vdx integrationem admittere, ita ut $\int Vdx$ futura sit certa ac definita functio quantitatum x, y, p, q, r, s etc. Vicissim autem quoties formula differentialis Vdx integrationem admittit eiusque propterea integrale $\int Vdx$ est vera functio quantitatum x, y, p, q, r, s etc., toties quoque eius variatio tantum ab extremis variationibus ipsarum x et y pendet neque variationes intermediae eam ullo modo afficere possunt. Ex quo necesse est, ut variationis pars integralis supra inventa evanescat, id quod fieri nequit, nisi fuerit

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0,$$

sicque theorema propositum etiam inversum veritati est consentaneum.[1])

COROLLARIUM 1

93. En ergo insigne criterium, cuius ope formula differentialis duarum variabilium, cuiuscunque gradus differentialia in eam ingrediantur, diiudicari potest, utrum sit integrabilis necne. Multo latius ergo patet illo criterio satis noto, quo formularum differentialium primi gradus integrabilitas dignosci solet.

COROLLARIUM 2

94. Primo ergo si quantitas V sit tantum functio ipsarum x et y nullam differentialium rationem involvens, ut sit

$$dV = Mdx + Ndy,$$

tum formula differentialis Vdx integrationem non admittit, nisi sit $N = 0$, hoc est, nisi V sit functio ipsius x tantum; quod quidem per se est perspicuum.

COROLLARIUM 3

95. Proposita autem huiusmodi formula differentiali $vdx + udy$, ea cum forma Vdx ob $dy = pdx$ comparata dat $V = v + pu$ ideoque

1) Iam prius, scilicet sub finem Commentationis (297 indicis ENESTROEMIANI) *Analytica explicatio methodi maximorum et minimorum* (vide notam p. 375), EULERUS egregium hoc theorema examini Analystarum subiecerat confirmans eius veritatem ex principiis adhuc expositis haud difficulter demonstrari posse. Quae demonstratio hic peracta est. F. E.

$$M = \left(\frac{dv}{dx}\right) + p\left(\frac{du}{dx}\right), \quad N = \left(\frac{dv}{dy}\right) + p\left(\frac{du}{dy}\right) \quad \text{et} \quad P = u,$$

quandoquidem quantitates v et u nulla differentialia implicare sumuntur. Erit ergo

$$dP = du = dx\left(\frac{du}{dx}\right) + dy\left(\frac{du}{dy}\right).$$

Quare cum criterium integrabilitatis postulet, ut sit

$$N - \frac{dP}{dx} = 0,$$

erit pro hoc casu

$$\left(\frac{dv}{dy}\right) + p\left(\frac{du}{dy}\right) - \left(\frac{du}{dx}\right) - p\left(\frac{du}{dy}\right) = 0$$

seu

$$\left(\frac{dv}{dy}\right) = \left(\frac{du}{dx}\right),$$

quod est criterium iam vulgo cognitum.

SCHOLION 1

96. Demonstratio huius theorematis omnino est singularis, cum ex doctrina variationum sit petita, quae tamen ab hoc argumento prorsus est aliena; vix vero alia via patet ad eius demonstrationem pertingendi.[1] Tum vero hic accuratior cognitio functionum diligenter est animadvertenda, qua ostendimus formulam integralem $\int V dx$ neutiquam ut functionem quantitatum x, y, p, q, r etc. spectari posse, nisi revera integrationem admittat. Natura enim functionum semper hanc proprietatem habet adiunctam, ut, statim atque quantitatibus, quae eam ingrediuntur, valores determinati tribuuntur, ipsa functio ex iis formata determinatum adipiscatur valorem; veluti haec functio xy, si ponamus $x = 2$ et $y = 3$, valorem accipit $= 6$. Longe secus autem evenit in formula integrali $\int y dx$, cuius valor pro casu $x = 2$ et $y = 3$ neutiquam assignari potest, nisi inter y et x certa quaedam relatio statuatur; tum autem ea formula abit in functionem unicae variabilis. For-

1) Cf. § 104. Primam demonstrationem mere analyticam dedit EULERI amicus et adiutor A. I. LEXELL (1740—1784) in Commentatione: *De criteriis integrabilitatis formularum differentialium*, Novi comment. acad. sc. Petrop. 15 (1770), 1771, p. 127. F. E.

mularum ergo integralium, quae integrari nequeunt, natura sollicite a natura functionum distingui debet, cum functiones, statim atque quantitatibus variabilibus, ex quibus conflantur, determinati valores tribuuntur, ipsae quoque determinatos valores recipiant, etiamsi variabiles nullo modo a se invicem pendeant; quod minime evenit in formulis integralibus, quippe quarum determinatio omnes plane valores intermedios simul includit. Imprimis autem huic discrimini universa doctrina de maximis et minimis, ad quam hic attendimus, innitur, ubi formulas, quibus maximi minimive proprietas conciliari debet, necessario eiusmodi integrales esse oportet, quae per se integrationem non admittant.[1])

SCHOLION 2

97. Ad maiorem theorematis illustrationem eiusmodi formulam integralem $\int V dx$ consideremus, quae per se sit integrabilis, ponamusque exempli gratia

$$\int V dx = \frac{x\,dy}{y\,dx} = \frac{xp}{y},$$

ita ut sit

$$V = \frac{p}{y} - \frac{xpp}{yy} + \frac{xq}{y}$$

atque ideo haec formula differentialis

$$\left(\frac{p}{y} - \frac{xpp}{yy} + \frac{xq}{y}\right) dx$$

sit absolute integrabilis, ac videamus, an theorema nostrum hanc integrabilitatem declaret.

Quantitatem ergo V differentiemus et differentiali cum forma

$$dV = M dx + N dy + P dp + Q dq$$

comparato obtinebimus

$$M = \frac{-pp}{yy} + \frac{q}{y}, \quad N = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy}, \quad P = \frac{1}{y} - \frac{2xp}{yy} \quad \text{et} \quad Q = \frac{x}{y}.$$

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1) Idem iam observaverat EULERUS in opere *Methodus inveniendi lineas curvas* (vide notam p. 375), cap. I § 34, sed criterium integrabilitatis, quod inde deduci potest, tum non animadvertisse videtur. F. E.

Cum nunc secundum theorema fieri debeat

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0,$$

primo colligimus differentiando

$$\frac{dP}{dx} = \frac{-3p}{yy} + \frac{4xpp}{y^3} - \frac{2xq}{yy} \quad \text{et} \quad \frac{dQ}{dx} = \frac{1}{y} - \frac{xp}{yy},$$

tum vero

$$\frac{ddQ}{dx^2} = \frac{-2p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy}.$$

Ergo

$$\frac{dP}{dx} - \frac{ddQ}{dx^2} = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy},$$

cui valori quantitas N utique est aequalis.

SCHOLION 3

98. Ceterum quando formula differentialis Vdx integrationem per se admittit ideoque posito

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

secundum theorema est

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0,$$

hinc alia insignia consectaria deducuntur. Primo enim cum per dx multiplicando et integrando fiat

$$\int Ndx - P + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \text{etc.} = A,$$

patet etiam formulam Ndx absolute esse integrabilem. Deinde cum hinc porro fiat

$$\int dx\left(\int Ndx - P\right) + Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.} = Ax + B,$$

etiam formula

$$dx\left(\int Ndx - P\right)$$

integrationem admittit. Postea etiam simili modo integrabilis erit haec forma

$$dx\left(\int dx\left(\int N dx - P\right) + Q\right),$$

tum vero etiam haec

$$dx\left(\int dx\left(\int dx\left(\int N dx - P\right) + Q\right) - R\right)$$

et ita porro. Unde sequens theorema non minus notatu dignum et in praxi utilissimum colligimus.

THEOREMA 4

99. *Positis $dy = p dx$, $dp = q dx$, $dq = r dx$, $dr = s dx$ etc. si V eiusmodi fuerit functio ipsarum x, y, p, q, r, s etc., ut formula differentialis $V dx$ per se sit integrabilis, tum posito*

$$dV = M dx + N dy + P dp + Q dq + R dr + S ds + \text{etc.}$$

etiam sequentes formulae differentiales per se integrationem admittent:

I. *Formula $N dx$ erit per se integrabilis;*

tum posito $P - \int N dx = \mathfrak{P}$:

II. *Formula $\mathfrak{P} dx$ erit per se integrabilis;*

porro posito $Q - \int \mathfrak{P} dx = \mathfrak{Q}$:

III. *Formula $\mathfrak{Q} dx$ erit per se integrabilis;*

deinde posito $R - \int \mathfrak{Q} dx = \mathfrak{R}$:

IV. *Formula $\mathfrak{R} dx$ erit per se integrabilis;*

ulterius posito $S - \int \mathfrak{R} dx = \mathfrak{S}$:

V. *Formula $\mathfrak{S} dx$ erit per se integrabilis*

et ita porro.

DEMONSTRATIO

Huius theorematis veritas iam in praecedente paragrapho est evicta, unde simul patet, si omnes hae formulae integrationem admittant, etiam principalem $V dx$ absolute fore integrabilem.[1])

1) Quod hic patere dicitur, emendatione quadam eget; quae quomodo fieri possit, ex § 102 intelligitur. F. E.

COROLLARIUM 1

100. Cum V sit functio quantitatum

$$x, \quad y, \quad p=\frac{dy}{dx}, \quad q=\frac{dp}{dx}, \quad r=\frac{dq}{dx} \quad \text{etc.},$$

quantitates per differentiationem inde derivatae M, N, P, Q, R etc. etiam ita exhiberi possunt, ut sit

$$M=\left(\frac{dV}{dx}\right), \quad N=\left(\frac{dV}{dy}\right), \quad P=\left(\frac{dV}{dp}\right), \quad Q=\left(\frac{dV}{dq}\right) \quad \text{etc.},$$

unde ob primam formulam patet, si fuerit formula Vdx integrabilis, tum etiam formulam $\left(\frac{dV}{dy}\right)dx$ fore integrabilem.

COROLLARIUM 2

101. Deinde ergo quoque ob eandem rationem formula haec $\left(\frac{ddV}{dy^2}\right)dx$ hincque porro istae $\left(\frac{d^3V}{dy^3}\right)dx$, $\left(\frac{d^4V}{dy^4}\right)dx$ etc. omnes per se integrationem admittent.

COROLLARIUM 3

102. Quia tot tantum litterae P, Q, R etc. adsunt, quoti gradus differentialia in formula Vdx reperiuntur, et sequentes omnes evanescunt, litterae germanicae inde derivatae $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{S}$ etc. tandem evanescere vel in functiones solius quantitatis x abire debent, quia alioquin sequentes integrabilitates locum habere non possent.

EXEMPLUM

103. *Sit V eiusmodi functio, ut fiat*

$$\int Vdx=\frac{y(dx^2+dy^2)^{\frac{3}{2}}}{xdxddy}.$$

Factis substitutionibus

$$dy=pdx, \quad dp=qdx, \quad dq=rdx$$

pro hoc exemplo functio V ita exprimetur

$$V=\frac{p(1+pp)^{\frac{3}{2}}}{xq}-\frac{y(1+pp)^{\frac{3}{2}}}{xxq}+\frac{3yp\sqrt{(1+pp)}}{x}-\frac{yr(1+pp)^{\frac{3}{2}}}{xqq},$$

unde per differentiationem elicimus sequentes valores

$$N = -\frac{(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3p\sqrt{(1+pp)}}{x} - \frac{r(1+pp)^{\frac{3}{2}}}{xqq},$$

$$P = \frac{(1+4pp)\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypr\sqrt{(1+pp)}}{xqq},$$

$$Q = -\frac{p(1+pp)^{\frac{3}{2}}}{xqq} + \frac{y(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr(1+pp)^{\frac{3}{2}}}{xq^3},$$

$$R = -\frac{y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Iam igitur primo integrabilem esse oportet formulam Ndx seu

$$-\frac{dx(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3pdx\sqrt{(1+pp)}}{x} - \frac{dq(1+pp)^{\frac{3}{2}}}{xqq},$$

unde statim patet integrale hoc fore

$$\int Ndx = \frac{(1+pp)^{\frac{3}{2}}}{xq}.$$

Iam pro secunda formula hinc nanciscimur

$$\mathfrak{P} = P - \int Ndx = \frac{3pp\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypr\sqrt{(1+pp)}}{xqq}$$

ita ut integranda sit haec formula

$$\mathfrak{P}dx = \frac{3pdy\sqrt{(1+pp)}}{xq} - \frac{3ypdx\sqrt{(1+pp)}}{xxq} + \frac{3ydx(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypdq\sqrt{(1+pp)}}{xqq},$$

cuius integrale vel saltem eius pars ex postremo membro manifesto colligitur $\frac{3yp\sqrt{(1+pp)}}{xq}$; cuius differentiale cum totam formulam exhauriat, erit

$$\int \mathfrak{P}dx = \frac{3yp\sqrt{(1+pp)}}{xq}.$$

Nunc pro tertia formula habebimus

$$\mathfrak{Q} = Q - \int \mathfrak{P}dx = -\frac{p(1+pp)^{\frac{3}{2}}}{xqq} + \frac{y(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr(1+pp)^{\frac{3}{2}}}{xq^3} - \frac{3yp\sqrt{(1+pp)}}{xq},$$

unde per dx multiplicando ob $dx = \frac{dp}{q}$ in ultimo membro fit

$$\mathfrak{Q}\,dx = -\frac{dy(1+pp)^{\frac{3}{2}}}{xqq} + \frac{ydx(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2ydq(1+pp)^{\frac{3}{2}}}{xq^3} - \frac{3ypdp\sqrt{(1+pp)}}{xqq},$$

cuius penultimum membrum declarat integrale

$$\int \mathfrak{Q}\,dx = -\frac{y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Quarta porro formula ita erit comparata

$$\mathfrak{R} = R - \int \mathfrak{Q}\,dx = 0,$$

unde perspicuum est non solum hanc $\mathfrak{R}\,dx$, sed etiam sequentes omnes fore integrabiles.

SCHOLION

104. Theoremata haec eo pulcriora videntur, quod eorum demonstratio eiusmodi principio innititur, cuius ratio hinc prorsus est aliena, propterea quod in his veritatibus nullum amplius vestigium variationum apparet; ex quo nullum est dubium, quin demonstratio etiam ex alio fonte magis naturali hauriri queat [§ 96].

CAPUT IV

DE VARIATIONE FORMULARUM INTEGRALIUM COMPLICATARUM DUAS VARIABILES INVOLVENTIUM

PROBLEMA 8

105. *Posito* $v = \int \mathfrak{V} dx$ *existente* $\mathfrak{V}$ *functione quacunque binarum variabilium* x, y *earumque differentialium*

$$dy = pdx, \quad dp = qdx, \quad dq = rdx \quad \text{etc.}$$

si V *denotet functionem quamcunque ipsius* v, *investigare variationem formulae integralis complicatae* $\int Vdx$.

SOLUTIO

Quia quantitas v ipsa est formula integralis $\int \mathfrak{V} dx$, formula $\int Vdx$ est utique complicata. Cum igitur functio V solam quantitatem v involvere ponatur, statuamus $dV = Ldv$; tum vero pro functione $\mathfrak{V}$ sit eius differentiale

$$d\mathfrak{V} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

His positis cum variatio quaesita sit

$$\delta \int Vdx = \int \delta(Vdx) = \int (\delta V dx + V d\delta x),$$

est per reductionem supra [§ 77] adhibitam

$$\delta \int Vdx = V\delta x + \int (dx\delta V - dV\delta x).$$

Cum autem per hypothesin sit $dV = Ldv$, erit etiam pro variatione

$\delta V = L\delta v$; verum ob $v = \int \mathfrak{V}dx$ erit primo $dv = \mathfrak{V}dx$ ideoque $dV = L\mathfrak{V}dx$, tum vero

$$\delta v = \delta \int \mathfrak{V}dx = \mathfrak{V}\delta x + \int (dx\delta\mathfrak{V} - d\mathfrak{V}\delta x)$$

ac propterea

$$\delta V = L\mathfrak{V}\delta x + L\int (dx\delta\mathfrak{V} - d\mathfrak{V}\delta x)$$

hincque

$$\delta \int Vdx = V\delta x + \int \left(L\mathfrak{V}dx\delta x + Ldx \int (dx\delta\mathfrak{V} - d\mathfrak{V}\delta x) - L\mathfrak{V}dx\delta x\right)$$

seu

$$\delta \int Vdx = V\delta x + \int Ldx \int (dx\delta\mathfrak{V} - d\mathfrak{V}\delta x).$$

Ex praecedente autem capite [§ 86] patet esse

$$\begin{aligned}\int (dx\delta\mathfrak{V} - d\mathfrak{V}\delta x) &= \delta \int \mathfrak{V}dx - \mathfrak{V}\delta x \\ &= \int \omega dx \left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\right) \\ &\quad + \omega \left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \frac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right) \\ &\quad + \frac{d\omega}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \frac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right) \\ &\quad + \frac{dd\omega}{dx^2}\left(\mathfrak{R} - \frac{d\mathfrak{S}}{dx} + \text{etc.}\right) \\ &\quad + \text{etc.}\end{aligned}$$

sumto elemento dx constante et posito brevitatis ergo $\omega = \delta y - p\delta x$.

Verum cum hinc substitutio molestias pariat, praestabit ex primo fonte rem repetere. Cum igitur ex differentiali et variatione quantitatis $\mathfrak{V}$ fiat

$$\begin{aligned}dx\delta\mathfrak{V} - d\mathfrak{V}\delta x = dx(&\mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}) \\ - \delta x(&\mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}),\end{aligned}$$

ob

$$dy = pdx, \quad dp = qdx, \quad dq = rdx, \quad dr = sdx \quad \text{etc.}$$

erit

$$dx\delta\mathfrak{V} - d\mathfrak{V}\delta x = \mathfrak{N}dx(\delta y - p\delta x) + \mathfrak{P}dx(\delta p - q\delta x) + \mathfrak{Q}dx(\delta q - r\delta x) + \text{etc.}$$

Verum ob dx constans ex § 79 fit

$$\delta y - p\delta x = \omega, \quad \delta p - q\delta x = \frac{d\omega}{dx}, \quad \delta q - r\delta x = \frac{dd\omega}{dx^2}, \quad \delta r - s\delta x = \frac{d^3\omega}{dx^3} \quad \text{etc.}$$

sicque habebitur

$$dx\delta\mathfrak{V} - d\mathfrak{V}\delta x = \mathfrak{N}\omega dx + \mathfrak{P}d\omega + \mathfrak{Q}\frac{dd\omega}{dx} + \mathfrak{R}\frac{d^3\omega}{dx^2} + \mathfrak{S}\frac{d^4\omega}{dx^3} + \text{etc.},$$

cuius quidem integrale praebet superiorem expressionem. Ponatur nunc integrale $\int Ldx = I$ eritque

$$\delta\int Vdx = V\delta x + I\int(dx\delta\mathfrak{V} - d\mathfrak{V}\delta x) - \int I(dx\delta\mathfrak{V} - d\mathfrak{V}\delta x).$$

Nunc vero facile [§ 81—85] colligitur fore

$$\begin{aligned}\int I(dx\delta\mathfrak{V} - d\mathfrak{V}\delta x) = \int\omega dx\Big(I\mathfrak{N} - \frac{d.I\mathfrak{P}}{dx} + \frac{dd.I\mathfrak{Q}}{dx^2} - \frac{d^3.I\mathfrak{R}}{dx^3} + \text{etc.}\Big)&\\ +\ \omega\Big(I\mathfrak{P} - \frac{d.I\mathfrak{Q}}{dx} + \frac{dd.I\mathfrak{R}}{dx^2} - \text{etc.}\Big)&\\ + \frac{d\omega}{dx}\Big(I\mathfrak{Q} - \frac{d.I\mathfrak{R}}{dx} + \text{etc.}\Big)&\\ + \text{etc.},&\end{aligned}$$

unde facta substitutione concluditur variatio quaesita

$$\begin{aligned}\delta\int Vdx = V\delta x + I\int\omega dx\Big(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \text{etc.}\Big)&\\ - \int\omega dx\Big(I\mathfrak{N} - \frac{d.I\mathfrak{P}}{dx} + \frac{dd.I\mathfrak{Q}}{dx^2} - \frac{d^3.I\mathfrak{R}}{dx^3} + \text{etc.}\Big)&\\ + I\omega\Big(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \frac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\Big)&\\ - \omega\Big(I\mathfrak{P} - \frac{d.I\mathfrak{Q}}{dx} + \frac{dd.I\mathfrak{R}}{dx^2} - \frac{d^3.I\mathfrak{S}}{dx^3} + \text{etc.}\Big)&\\ + \frac{Id\omega}{dx}\Big(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \frac{dd\mathfrak{S}}{dx^2} - \text{etc.}\Big)&\\ - \frac{d\omega}{dx}\Big(I\mathfrak{Q} - \frac{d.I\mathfrak{R}}{dx} + \frac{dd.I\mathfrak{S}}{dx^2} - \text{etc.}\Big)&\\ + \frac{Idd\omega}{dx^2}\Big(\mathfrak{R} - \frac{d\mathfrak{S}}{dx} + \text{etc.}\Big)&\\ - \frac{dd\omega}{dx^2}\Big(I\mathfrak{R} - \frac{d.I\mathfrak{S}}{dx} + \text{etc.}\Big)&\\ + \frac{Id^3\omega}{dx^3}(\mathfrak{S} - \text{etc.})&\\ - \frac{d^3\omega}{dx^3}(I\mathfrak{S} - \text{etc.})&\\ + \text{etc.}&\end{aligned}$$

Si hic partes binae priores differentiatae iterum integrentur, reliquarum facta reductione impetrabimus loco dI valorem Ldx restituendo

$$\delta\int Vdx = V\delta x + \int Ldx \int \omega dx \left(\mathfrak{R} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \text{etc.}\right)$$
$$+ \int \omega dx \left(L\mathfrak{P} - \frac{Ld\mathfrak{Q} + d.L\mathfrak{Q}}{dx} + \frac{Ldd\mathfrak{R} + d.Ld\mathfrak{R} + dd.L\mathfrak{R}}{dx^2} - \text{etc.}\right)$$
$$+ \omega \left(L\mathfrak{Q} - \frac{Ld\mathfrak{R} + d.L\mathfrak{R}}{dx} + \frac{Ldd\mathfrak{S} + d.Ld\mathfrak{S} + dd.L\mathfrak{S}}{dx^2} - \text{etc.}\right)$$
$$+ \frac{d\omega}{dx}\left(L\mathfrak{R} - \frac{Ld\mathfrak{S} + d.L\mathfrak{S}}{dx} + \text{etc.}\right)$$
$$+ \frac{dd\omega}{dx^2}(L\mathfrak{S} - \text{etc.})$$
$$+ \text{etc.},$$

quae forma videtur simplicissima et ad usum maximo accommodata.

COROLLARIUM 1

106. Si eiusmodi relatio inter x et y quaeratur, ut integrale $\int Vdx$ maximum minimumve evadat, variationis partes integrales nihilo aequari oportet; quod in genere fieri nequit, sed ad terminum, quousque integrale $\int Vdx$ extenditur, spectari oportet; pro quo si ponamus fieri $I = \int Ldx = A$, ex priori forma colligimus hanc aequationem

$$0 = (A - I)\mathfrak{R} - \frac{d.(A-I)\mathfrak{P}}{dx} + \frac{dd.(A-I)\mathfrak{Q}}{dx^2} - \frac{d^3.(A-I)\mathfrak{R}}{dx^3} + \text{etc.}$$

COROLLARIUM 2

107. Quomodocunque autem haec aequatio pro quovis casu oblato tractetur, semper tandem eo est deveniendum, ut formula integralis $I = \int Ldx$ per differentiationem exturbari debeat, qua operatione simul quantitatem A inde extrudi evidens est; sicque aequatio resultans non amplius a termino integrationis pendebit.

COROLLARIUM 3

108. Quodsi in genere pro variatione formulae integralis $\int Vdx$ invenienda valorem $\int Ldx = I$ toti integrali respondentem ponamus $= A$, va-

riatio quaesita ita exprimetur

$$\delta\int Vdx = V\delta x + \int\omega dx\Big((A-I)\mathfrak{N} - \frac{d.(A-I)\mathfrak{P}}{dx} + \frac{dd.(A-I)\mathfrak{Q}}{dx^2} - \frac{d^3.(A-I)\mathfrak{R}}{dx^3} + \text{etc.}\Big)$$

$$+\,\omega\Big(L\mathfrak{Q} - \frac{Ld\mathfrak{R} + d.L\mathfrak{R}}{dx} + \frac{Ldd\mathfrak{S} + d.Ld\mathfrak{S} + dd.L\mathfrak{S}}{dx^2} - \text{etc.}\Big)$$

$$+\,\frac{d\omega}{dx}\Big(L\mathfrak{R} - \frac{Ld\mathfrak{S} + d.L\mathfrak{S}}{dx} + \text{etc.}\Big)$$

$$+\,\frac{dd\omega}{dx^2}(L\mathfrak{S} - \text{etc.})$$

$$+\,\text{etc.},$$

ubi $A - I$ est valor formulae $\int Ldx$ a termino integrationis extremo ad quemvis locum indefinitum medium retro sumtus.

SCHOLION

109. In solutione huius problematis compendium se obtulit, quo etiam analysis in superiori capite adhibita non mediocriter contrahi potest. Cum enim ibi (§ 79) pervenissemus ad

$$\delta\int Vdx = V\delta x + \int(dx\delta V - dV\delta x),$$

ob

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

et

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}$$

erit

$$dV = dx(M + Np + Pq + Qr + Rs + \text{etc.})$$

hincque colligitur

$$dx\delta V - dV\delta x = dx\big(N(\delta y - p\delta x) + P(\delta p - q\delta x) + Q(\delta q - r\delta x) + \text{etc.}\big).$$

Iam si brevitatis gratia ponatur $\delta y - p\delta x = \omega$, erit differentiando

$$\delta(pdx) - qdx\delta x - p\delta dx = d\omega;$$

at

$$\delta(pdx) = pd\delta x + \delta pdx,$$

ergo

$$\delta p - q\delta x = \frac{d\omega}{dx}.$$

Simili modo hanc formulam differentiando ob $dp = qdx$ et $dq = rdx$ fit

$$qd\delta x + \delta q dx - qd\delta x - dq\delta x = dx(\delta q - r\delta x) = d.\frac{d\omega}{dx},$$

unde perspicuum est posito $\delta y - p\delta x = \omega$ fore

$$\delta p - q\delta x = \frac{d\omega}{dx}, \quad \delta q - r\delta x = \frac{1}{dx}d.\frac{d\omega}{dx} = \frac{dd\omega}{dx^2}$$

sumto dx constante,

$$\delta r - s\delta x = \frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx} = \frac{d^3\omega}{dx^3}$$

etc.

Quocirca erit

$$dx\delta V - dV\delta x = dx\left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \frac{Sd^4\omega}{dx^4} + \text{etc.}\right),$$

siquidem differentiale dx constans accipiatur.

PROBLEMA 9

110. *Si fuerit* $v = \int \mathfrak{V}dx$ *existente*

$$d\mathfrak{V} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.},$$

tum vero sit V *functio quaecunque non solum quantitates*

$$x, \quad y, \quad p = \frac{dy}{dx}, \quad q = \frac{dp}{dx}, \quad r = \frac{dq}{dx} \quad etc.,$$

sed etiam ipsam formulam integralem $v = \int \mathfrak{V}dx$ *implicans, investigare variationem formulae integralis complicatae* $\int Vdx$.

SOLUTIO

Quoniam V est functio quantitatum v, x, y, p, q, r etc., sumatur eius differentiale, quod sit

$$dV = Ldv + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

ac habebitur variatio ipsius V ita expressa

$$\delta V = L\delta v + M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.};$$

tum vero notetur ob

$$dv = \mathfrak{V}dx, \quad dy = pdx, \quad dp = qdx \quad \text{etc.}$$

esse

$$dV = dx(L\mathfrak{V} + M + Np + Pq + Qr + Rs + \text{etc.})$$

et

$$\delta\mathfrak{V} = \mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}$$

Praeterea habemus

$$\delta v = \int(\mathfrak{V}d\delta x + dx\delta\mathfrak{V}) = \mathfrak{V}\delta x + \int(dx\delta\mathfrak{V} - d\mathfrak{V}\delta x),$$

unde posito $\delta y - p\delta x = \omega$ erit per ante inventa

$$\delta v = \mathfrak{V}\delta x + \int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right),$$

ubi commoditatis ergo sumsimus dx constans.

His praeparatis, cum variatio quaesita sit

$$\delta\int Vdx = V\delta x + \int(dx\delta V - dV\delta x),$$

ut reductione supra inventa uti possimus, ponamus

$$dV = Ldv + dW,$$

ut sit

$$\delta V = L\delta v + \delta W \quad \text{et} \quad dW = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

Quocirca nanciscemur hanc formam

$$\delta\int Vdx = V\delta x + \int(Ldx\delta v - Ldv\delta x) + \int(dx\delta W - dW\delta x),$$

ubi est

$$dx\delta W - dW\delta x = dx\left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \text{etc.}\right).$$

Tum vero est

$$dx\delta v - dv\delta x = dx\int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right)$$

ob $dv\delta x = \mathfrak{V}dx\delta x$. Quibus substitutis colligitur variatio quaesita

$$\delta\int Vdx = V\delta x + \int Ldx\int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right)$$
$$+\int dx\left(N + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \text{etc.}\right).$$

Quo iam hanc formam ulterius reducamus, ponamus integrale $\int Ldx = I$ ita sumtum, ut pro initio, unde integrale $\int Vdx$ capitur, evanescat, pro fine autem, ubi integrale $\int Vdx$ terminatur, fiat $I = A$; sicque fiet

$$\delta\int Vdx = V\delta x + A\int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right)$$
$$-\int Idx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right)$$
$$+\int dx\left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \text{etc.}\right),$$

ad quam formam contrahendam statuamus

$$N + (A - I)\mathfrak{N} = N',$$
$$P + (A - I)\mathfrak{P} = P',$$
$$Q + (A - I)\mathfrak{Q} = Q',$$
$$R + (A - I)\mathfrak{R} = R'$$
$$\text{etc.},$$

ut prodeat forma illi, quam supra [§ 79—85] tractavimus, similis

$$\delta\int Vdx = V\delta x + \int dx\left(N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.}\right);$$

ubi ergo si post signum integrale differentialia ipsius ω eliminentur, perveniemus secundum § 86 ad hanc expressionem

$$\begin{aligned}\delta\int Vdx = \int \omega\, dx\Big(N' - \frac{dP'}{dx} + \frac{ddQ'}{dx^2} - \frac{d^3R'}{dx^3} + \frac{d^4S'}{dx^4} - \text{etc.}\Big)\\ + V\delta x + \omega\Big(P' - \frac{dQ'}{dx} + \frac{ddR'}{dx^2} - \frac{d^3S'}{dx^3} + \text{etc.}\Big)\\ + \text{Const.} + \frac{d\omega}{dx}\Big(Q' - \frac{dR'}{dx} + \frac{ddS'}{dx^2} - \text{etc.}\Big)\\ + \frac{dd\omega}{dx^2}\Big(R' - \frac{dS'}{dx} + \text{etc.}\Big)\\ + \frac{d^3\omega}{dx^3}(S' - \text{etc.})\\ + \text{etc.}\end{aligned}$$

Constanti autem per integrationem invectae eiusmodi valor tribui debet, ut pro initio integrationis formulae $\delta\int Vdx$ partes absolutae ad nihilum redigantur, siquidem prima pars integralis ita sumatur, ut pro eodem initio evanescat; tum vero universam expressionem ad finem integrationis produci oportet, pro quo iam posuimus fieri $\int Ldx = I = A$.

COROLLARIUM 1

111. In parte integrali variabilitas per totam integrationis extensionem debet comprehendi, in partibus autem absolutis sufficit respexisse ad initium ac finem integrationis, pro utroque autem termino conditiones variationis praescriptae suppeditant valores δx, ω, $\frac{d\omega}{dx}$, $\frac{dd\omega}{dx^2}$ etc. Ac postquam ex conditionibus initii constans rite fuerit determinata, tum superest, ut singula membra ad finem integrationis accommodentur.

COROLLARIUM 2

112. Pro initio igitur integrationis, ubi $I = 0$, erit primo

$$N' = N + A\mathfrak{N},\quad P' = P + A\mathfrak{P},\quad Q' = Q + A\mathfrak{Q},\quad R' = R + A\mathfrak{R}\quad \text{etc.},$$

pro differentialibus vero ob $dI = Ldx$ erit

$$\frac{dN'}{dx} = \frac{dN}{dx} + \frac{Ad\mathfrak{N}}{dx} - L\mathfrak{N}$$

et ita de reliquis similique modo pro differentialibus secundis

$$\frac{ddN'}{dx^2} = \frac{ddN}{dx^2} + \frac{A\,dd\mathfrak{N}}{dx^2} - \frac{2\,L\,d\mathfrak{N}}{dx} - \frac{\mathfrak{N}\,dL}{dx}.$$

COROLLARIUM 3

113. Pro fine autem integrationis, ubi $I = A$, fit

$$N' = N, \quad P' = P, \quad Q' = Q, \quad R' = R \quad \text{etc.},$$

valores vero differentiales ita se habebunt

$$\frac{dN'}{dx} = \frac{dN}{dx} - L\mathfrak{N}, \quad \frac{dP'}{dx} = \frac{dP}{dx} - L\mathfrak{P}, \quad \frac{dQ'}{dx} = \frac{dQ}{dx} - L\mathfrak{Q} \quad \text{etc.},$$

secundi vero gradus hoc modo

$$\frac{ddN'}{dx^2} = \frac{ddN}{dx^2} - \frac{2\,L\,d\mathfrak{N}}{dx} - \frac{\mathfrak{N}\,dL}{dx},$$

$$\frac{ddP'}{dx^2} = \frac{ddP}{dx^2} - \frac{2\,L\,d\mathfrak{P}}{dx} - \frac{\mathfrak{P}\,dL}{dx}$$

et ita porro.

SCHOLION 1

114. Quanquam natura variationum atque etiam quaestionum eo pertinentium iam satis est explicata, tamen huius argumenti tam dignitas quam novitas ampliorem illustrationem requirere videntur, cum ne superfluum quidem foret eadem saepius inculcari. Cum igitur ante[1]) Geometria et huius calculi applicatione ad maxima et minima usi simus, ad hanc doctrinam magis explanandam hic rem generalius pro sola Analysi contemplabimur.

Primo igitur spectatur relatio quaecunque inter binas variabiles x et y, sive ea sit cognita sive demum definienda, indeque formata consideratur formula integralis quaecunque $\int V dx$, quae intra certos terminos comprehensa seu integratione a dato initio ad datum finem extensa utique certum quendam valorem recipere debet. Tum illa relatio inter x et y, quaecunque fuerit, quomodocunque infinite parum immutetur, ut singulis x variationibus quibuscunque δx auctis iam respondeant eaedem y variationibus quoque quibuscunque δy auctae, ubi quidem observandum est tam in initio quam

1) *Methodus inveniendi lineas curvas*; vide notam p. 375. F. E.

fine rationem harum variationum per conditiones quaestionum dari, in medio autem istas variationes ita generaliter assumi, ut nulla plane lege inter se connectantur. Tum ex hac relatione variata eiusdem formulae integralis $\int V dx$ [valor] ab eodem initio ad eundem finem expansus seu intra eosdem terminos contentus definiri concipitur ac tota iam quaestio in hoc versatur, ut huius postremi valoris variati excessus supra priorem illum valorem formulae $\int V dx$ investigetur. Qui excessus cum per $\delta \int V dx$, quae forma ipsa est variatio formulae $\int V dx$, indicetur, huius quaestionis solutionem hactenus dedimus ita late patentem, ut omnes casus, quibus quantitas V est functio quaecunque non solum ipsarum x, y, p, q, r, s etc., sed etiam insuper formulam quandam integralem $v = \int \mathfrak{V} dx$ utcunque involvens, in se complectatur.

SCHOLION 2

115. Quod in praecedente capite tacite assumsimus de quantitate constante variationi inventae adiicienda, quippe quam pars integralis variationis sponte involvit, hoc in istius problematis solutione accuratius exponere est visum. Cum scilicet in huiusmodi quaestionibus, quae ad formulas integrales reducuntur, perpetuo ad terminos integrationis sit respiciendum, siquidem integrale nihil aliud est nisi summa elementorum a termino dato seu initio ad alium terminum seu finem continuatorum, haec consideratio prorsus essentialis est omni integrationi, sine qua idea valoris integralis ne consistere quidem potest. Quamobrem constitutis integrationis terminis, initio scilicet et fine, statim ac variationis pars integralis ita est accepta, ut pro initio evadat nulla, tum eiusmodi constantem adiici oportet, ut etiam partes absolutae pro eodem initio destruantur sicque universa variationis expressio ad nihilum redigatur. Quod cum fuerit factum, ad finem integrationis demum progredi licet, ut hoc pacto vera variatio formulae integralis propositae ab initio ad finem extensae obtineatur.

Haec autem variationum doctrina ad duplicis generis quaestiones accommodari potest; dum in altero relatio inter variabiles x et y data assumitur et formulae integralis itidem datae $\int V dx$ variatio investigatur, postquam per totam integrationis extensionem variabilibus x et y variationes quaecunque fuerint tributae, in altero autem genere ipsa illa variabilium x et y relatio quaeritur, ut formulae integralis $\int V dx$ variatio certa proprietate sit praedita; quemadmodum si ea formula maximum minimumve valorem recipere debeat, hanc variationem in nihilum abire necesse est. Ubi iterum duo casus

se offerunt, prout maximum minimumve locum habere debet, vel quaecunque variationes ipsis x et y tribuantur, vel si tantum hae variationes certae cuidam legi adstringantur. Ex quo manifestum est hanc theoriam multo latius patere, quam quidem ea adhuc in usum est vocata.

PROBLEMA 10

116. *Si functio V praeter binas variabiles x, y cum suis valoribus differentialibus*

$$p = \frac{dy}{dx}, \quad q = \frac{dp}{dx}, \quad r = \frac{dq}{dx} \quad \textit{etc.}$$

etiam duas pluresve formulas integrales

$$v = \int \mathfrak{V} dx, \quad v' = \int \mathfrak{V}' dx \quad \textit{etc.}$$

involvat, ut sit

$$d\mathfrak{V} = \mathfrak{M} dx + \mathfrak{N} dy + \mathfrak{P} dp + \mathfrak{Q} dq + \mathfrak{R} dr + \text{etc.},$$

$$d\mathfrak{V}' = \mathfrak{M}' dx + \mathfrak{N}' dy + \mathfrak{P}' dp + \mathfrak{Q}' dq + \mathfrak{R}' dr + \text{etc.}$$

atque differentiali sumto

$$dV = L dv + L' dv' + M dx + N dy + P dp + Q dq + \text{etc.},$$

invenire variationem formulae integralis $\int V dx$.

SOLUTIO

Si huius problematis solutio eodem modo instituatur ac praecedentis, mox patebit calculum a geminata formula integrali

$$v = \int \mathfrak{V} dx \quad \text{et} \quad v' = \int \mathfrak{V}' dx$$

non turbari neque etiam, si plures eiusmodi involverentur. Quare tota solutio tandem huc redibit, ut constitutis integrationis terminis primo integralia

$$\int L dx = I \quad \text{et} \quad \int L' dx = I'$$

ita sint capienda, ut pro initio integrationis evanescant, tum vero pro fine

integrationis fiat $I = A$ et $I' = A'$; quibus quantitatibus inventis statuatur porro

$$N + (A - I)\mathfrak{N} + (A' - I')\mathfrak{N}' = N',$$
$$P + (A - I)\mathfrak{P} + (A' - I')\mathfrak{P}' = P',$$
$$Q + (A - I)\mathfrak{Q} + (A' - I')\mathfrak{Q}' = Q',$$
$$R + (A - I)\mathfrak{R} + (A' - I')\mathfrak{R}' = R'$$
etc.

eritque variatio quaesita, dum utrique variabili x et y variationes quaecunque tribuuntur, ex praecedentibus solutionibus [§ 110]

$$\begin{aligned}\delta\int Vdx = \int \omega dx\left(N' - \frac{dP'}{dx} + \frac{ddQ'}{dx^2} - \frac{d^3R'}{dx^3} + \frac{d^4S'}{dx^4} - \text{etc.}\right)\\
+ V\delta x + \omega\left(P' - \frac{dQ'}{dx} + \frac{ddR'}{dx^2} - \frac{d^3S'}{dx^3} + \text{etc.}\right)\\
+ \text{Const.} + \frac{d\omega}{dx}\left(Q' - \frac{dR'}{dx} + \frac{ddS'}{dx^2} - \text{etc.}\right)\\
+ \frac{dd\omega}{dx^2}\left(R' - \frac{dS'}{dx} + \text{etc.}\right)\\
+ \frac{d^3\omega}{dx^3}(S' - \text{etc.})\\
+ \text{etc.},\end{aligned}$$

ubi commoditatis gratia elementum dx constans est assumtum.

COROLLARIUM

117. Si ergo etiam plures huiusmodi formulae integrales $\int \mathfrak{B}dx$ in functionem V quomodocunque ingrediantur, expressio variationis quaesitae inde non mutatur, sed tantum quantitates N', P', Q', R' etc. ex iis rite definiri convenit.

SCHOLION

118. Etsi formulae integrales

$$I = \int Ldx, \quad I' = \int L'dx$$

binas variabiles involvunt ideoque valores fixos recipere non posse videntur,

tamen perpendendum est in omnibus huiusmodi quaestionibus semper certam quandam relationem inter binas variabiles x et y supponi, sive ea absolute detur sive demum per calculum definiri debeat. Hac igitur ipsa relatione iam in usum vocata, ut quantitas y instar functionis ipsius x spectari possit, formulae illae integrales utique determinatos valores sortientur.

PROBLEMA 11

119. *Si functio* $\mathfrak{V}$ *praeter variabiles* x *et* y *earumque valores differentiales* p, q, r, s *etc. ipsam quoque formulam integralem* $u = \int \mathfrak{v}\, dx$ *involvat, ut eius differentiale sit*

$$d\mathfrak{V} = \mathfrak{L}du + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc.}$$

existente

$$d\mathfrak{v} = \mathfrak{m}dx + \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + \text{etc.},$$

tum vero sit V *functio quaecunque ipsarum* x, y, p, q, r *etc. insuperque formulae integralis* $v = \int \mathfrak{V}\, dx$, *ut sit*

$$dV = Ldv + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

invenire variationem formulae integralis $\int V dx$.

SOLUTIO

Ex Problemate 9 statim invenimus variationem formulae integralis $\int \mathfrak{V} dx = v$; constitutis enim integrationis terminis sumtoque integrali $\int \mathfrak{L} dx = \mathfrak{J}$ ita, ut evanescat pro integrationis initio, pro fine fiat $\mathfrak{J} = \mathfrak{A}$, tum fiat brevitatis gratia

$$\mathfrak{N} + (\mathfrak{A} - \mathfrak{J})\mathfrak{n} = \mathfrak{N}', \quad \mathfrak{P} + (\mathfrak{A} - \mathfrak{J})\mathfrak{p} = \mathfrak{P}', \quad \mathfrak{Q} + (\mathfrak{A} - \mathfrak{J})\mathfrak{q} = \mathfrak{Q}' \quad \text{etc.};$$

erit ex illius problematis solutione

$$\delta v = \mathfrak{V}\delta x + \int dx \left(\mathfrak{N}'\omega + \frac{\mathfrak{P}' d\omega}{dx} + \frac{\mathfrak{Q}' dd\omega}{dx^2} + \frac{\mathfrak{R}' d^3\omega}{dx^3} + \text{etc.}\right)$$

posito $\omega = \delta y - p\delta x$ et sumto dx constante.

Iam vero cum quaeratur $\delta \int V dx$, ob

$$\delta \int V dx = V\delta x + \int (dx\delta V - dV\delta x)$$

posito brevitatis ergo

$$dV = Ldv + dW \quad \text{et} \quad \delta V = L\delta v + \delta W,$$

ut sit

$$dW = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

erit, ut ibidem vidimus,

$$\delta\int Vdx = V\delta x + \int(Ldx\delta v - Ldv\delta x)$$
$$+\int dx\left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \text{etc.}\right);$$

ubi si loco dv et δv valores modo inventi substituantur, erit

$$dx\delta v - dv\delta x = dx\int dx\left(\mathfrak{N}'\omega + \frac{\mathfrak{P}'d\omega}{dx} + \frac{\mathfrak{Q}'dd\omega}{dx^2} + \frac{\mathfrak{R}'d^3\omega}{dx^3} + \text{etc.}\right).$$

Nunc ponatur $\int Ldx = I$ integrali ita sumto, ut evanescat in integrationis initio, in fine autem fiat $I = A$, et habebimus

$$\int L(dx\delta v - dv\delta x) = \int(A-I)dx\left(\mathfrak{N}'\omega + \frac{\mathfrak{P}'d\omega}{dx} + \frac{\mathfrak{Q}'dd\omega}{dx^2} + \frac{\mathfrak{R}'d^3\omega}{dx^3} + \text{etc.}\right).$$

Restituantur pro $\mathfrak{N}'$, $\mathfrak{P}'$, $\mathfrak{Q}'$, $\mathfrak{R}'$ etc. valores supra assumti et ad calculum contrahendum ponatur

$$N + (A-I)\mathfrak{N} + (A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{n} = N',$$
$$P + (A-I)\mathfrak{P} + (A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{p} = P',$$
$$Q + (A-I)\mathfrak{Q} + (A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{q} = Q',$$
$$R + (A-I)\mathfrak{R} + (A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{r} = R'$$
$$\text{etc.}$$

ac manifestum est fore variationem quaesitam

$$\delta\int Vdx = V\delta x + \int dx\left(N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.}\right),$$

quae forma porro evolvitur in eandem expressionem, quam sub finem Problematis 9 (§ 110) exhibuimus, quam ergo hic denuo apponere foret superfluum.

COROLLARIUM 1

120. Hic ergo formula integralis $\int Vdx$, cuius variationem assignavimus, ita est comparata, ut non solum functio V formulam integralem $\int \mathfrak{B}dx$ in-

volvat, sed etiam haec functio $\mathfrak{V}$ aliam formulam integralem $\int \mathfrak{v}\,dx$ in se complectatur, ubi quidem functio $\mathfrak{v}$ nullam amplius formulam integralem implicat.

COROLLARIUM 2

121. Sin autem et haec functio $\mathfrak{v}$ insuper formulam integralem in se involvat, iam satis perspicuum est, quomodo tum solutionem institui oporteat, siquidem tum valores N', P', Q', R' etc. partes insuper recipient a postrema formula integrali pendentes.

SCHOLION

122. Quomodocunque ergo formula integralis $\int V dx$ fuerit complicata, praecepta hactenus exposita omnino sufficiunt ad eius variationem investigandam, etiamsi forte complicatio fuerit infinita. Cum igitur omnes expressiones binas variabiles implicantes, quarum variationes unquam sint investigandae, vel a formulis integralibus sint liberae vel unam pluresve in se complectantur easque vel simplices vel complicatas utcunque, huic calculi variationum parti, quae circa duas variabiles versatur, abunde satisfactum videtur, ut vix quicquam amplius desiderari queat. Quamobrem ad formulas trium variabilium progrediamur ac primo quidem tales, quarum relatio per geminam aequationem definiri ponitur, ut binae variabiles tanquam functiones tertiae spectari queant, sive haec duplex relatio sit cognita sive ex ipsa variationis indole investiganda.

CAPUT V

DE VARIATIONE FORMULARUM INTEGRALIUM TRES VARIABILES INVOLVENTIUM ET DUPLICEM RELATIONEM IMPLICANTIUM

PROBLEMA 12

123. *Proposita formula quacunque ternas variabiles x, y, z cum suis differentialibus cuiuscunque gradus involvente eius variationem definire ex variationibus omnium trium variabilium oriundam.*

SOLUTIO

Sit W formula ista proposita, cuius primo quaeratur valor variatus $W+\delta W$, qui oritur, si loco x, y, z scribantur ipsarum valores variati

$$x+\delta x, \quad y+\delta y, \quad z+\delta z$$

similiterque pro earum differentialibus

$$dx+d\delta x, \quad dy+d\delta y, \quad dz+d\delta z$$

et ita porro; a quo si ipsa formula W auferatur, remanebit eius variatio δW. Ex quo intelligitur hanc variationem per consuetam differentiationem obtineri, si modo loco signi differentiationis d signum variationis δ scribatur. Tantum notasse iuvabit, si differentialium variationes capi oporteat, perinde esse, in quonam loco inter differentiationis signa signum variationis δ collocetur, quemadmodum supra [§ 37, 40] demonstravimus; unde signum variationis per-

petuo in postremo loco poni poterit, quod, cum ad formulas integrales progrediemur, commodissimum videtur, sicut ex iis, quae hactenus de formulis integralibus binas variabiles involventibus sunt tradita, satis est manifestum.

COROLLARIUM 1

124. Quoniam z perinde ac y tanquam functio ipsius x spectari potest, si ponatur $\frac{dy}{dx} = p$ et $\frac{dz}{dx} = \mathfrak{p}$, erit

$$\delta p = \frac{d\delta y - p\,d\delta x}{dx} \quad \text{et} \quad \delta \mathfrak{p} = \frac{d\delta z - \mathfrak{p}\,d\delta x}{dx}$$

similique modo formulae hinc derivatae a superioribus non discrepant.

COROLLARIUM 2

125. Ponamus $\delta y - p\delta x = \omega$ et $\delta z - \mathfrak{p}\delta x = \mathfrak{w}$ eritque

$$d\delta y - pd\delta x - qdx\delta x = d\omega \quad \text{et} \quad d\delta z - \mathfrak{p}d\delta x - \mathfrak{q}dx\delta x = d\mathfrak{w},$$

si scilicet statuamus

$$\frac{dp}{dx} = q \quad \text{et} \quad \frac{d\mathfrak{p}}{dx} = \mathfrak{q},$$

unde patet fore

$$\delta p - q\delta x = \frac{d\omega}{dx} \quad \text{et} \quad \delta\mathfrak{p} - \mathfrak{q}\delta x = \frac{d\mathfrak{w}}{dx}.$$

COROLLARIUM 3

126. Si ulterius statuamus

$$\frac{dq}{dx} = r, \quad \frac{d\mathfrak{q}}{dx} = \mathfrak{r}, \quad \frac{dr}{dx} = s, \quad \frac{d\mathfrak{r}}{dx} = \mathfrak{s} \quad \text{etc.},$$

erit simili modo sumto dx constante

$$\delta q - r\delta x = \frac{dd\omega}{dx^2}, \quad \delta\mathfrak{q} - \mathfrak{r}\delta x = \frac{dd\mathfrak{w}}{dx^2},$$

$$\delta r - s\delta x = \frac{d^3\omega}{dx^3}, \quad \delta\mathfrak{r} - \mathfrak{s}\delta x = \frac{d^3\mathfrak{w}}{dx^3}$$

sicque deinceps.

SCHOLION 1

127. Sive ergo formula varianda habuerit valorem finitum sive infinitum sive evanescentem, ope horum praeceptorum eius variatio perinde ac supra inveniri potest; neque enim haec praecepta a superioribus discrepant, nisi quod hic duplicis generis valores differentiales, alteri litteris latinis p, q, r, s etc., alteri germanicis $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ etc. indicati, introduci debeant; cuius rei ratio in eo est sita, quod hic utraque variabilis y et z tanquam functio ipsius x spectari potest. Sin autem unica aequatio inter ternas coordinatas daretur vel quaereretur, litterae hic introductae p et $\mathfrak{p}$ nullos habiturae essent valores certos, cum salva illa aequatione fractiones $\frac{dy}{dx}$ et $\frac{dz}{dx}$ omnes omnino valores recipere possent. Omissis autem his litteris ipsisque differentialibus in calculo relictis, etiam pro hoc casu regula in solutione exposita variationem declarabit.

SCHOLION 2

128. Supra [§ 12—24] iam notavi hunc casum trium variabilium, quarum relatio gemina aequatione definitur, sollicite esse distinguendum ab eo, ubi relatio unica aequatione definiri assumitur. Discrimen hoc ex Geometria clarissime illustratur, ubi ternae variabiles vicem ternarum coordinatarum gerunt; totidem autem in calculo adhiberi oportet, non solum quando quaestio circa superficies versatur, sed etiam quando lineae curvae non in eodem plano sitae sunt explorandae. Atque hoc quidem casu posteriori determinatio lineae curvae duas aequationes inter ternas coordinatas postulat, ita ut binae quaevis tanquam functiones tertiae spectari possint. Superficiei autem natura iam unica aequatione inter ternas coordinatas definitur, ita ut unaquaeque tanquam functio binarum reliquarum spectari queat, unde ingens discrimen in ipsa tractatione oritur. Praesens igitur caput inservire poterit eiusmodi lineis curvis indagandis, quae non in eodem plano sitae maximi minimive quapiam gaudeant proprietate.

PROBLEMA 13

129. *Si V fuerit functio quaecunque trium variabilium x, y, z, earum insuper differentialia cuiuscunque ordinis implicans, eaeque variabiles variationes quascunque recipiant, invenire variationem formulae integralis $\int V dx$.*

SOLUTIO

Quaecunque differentialia in functionem V ingrediantur, ea his factis substitutionibus

$$dy = p\,dx,\quad dp = q\,dx,\quad dq = r\,dx,\quad dr = s\,dx\quad \text{etc.},$$
$$dz = \mathfrak{p}\,dx,\quad d\mathfrak{p} = \mathfrak{q}\,dx,\quad d\mathfrak{q} = \mathfrak{r}\,dx,\quad d\mathfrak{r} = \mathfrak{s}\,dx\quad \text{etc.}$$

tollentur et quantitas V erit functio quantitatum finitarum x, y, z, p, q, r, s etc., $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}$ etc. Eius ergo differentiale huiusmodi habebit formam

$$\begin{aligned} dV = M\,dx + N\,dy + P\,dp + Q\,dq + R\,dr + S\,ds + \text{etc.}\\ + \mathfrak{N}\,dz + \mathfrak{P}\,d\mathfrak{p} + \mathfrak{Q}\,d\mathfrak{q} + \mathfrak{R}\,d\mathfrak{r} + \mathfrak{S}\,d\mathfrak{s} + \text{etc.},\end{aligned}$$

unde mutatis signis differentiationis d in δ simul habebitur variatio δV. Ex supra autem demonstratis etiam pro hoc casu trium variabilium habebitur

$$\delta\int V\,dx = \int(V\,d\delta x + dx\,\delta V) = V\delta x + \int(dx\,\delta V - dV\delta x).$$

At facta substitutione fiet

$$\begin{aligned}\frac{dx\,\delta V - dV\,\delta x}{dx} = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\\ + \mathfrak{N}\delta z + \mathfrak{P}\delta\mathfrak{p} + \mathfrak{Q}\delta\mathfrak{q} + \mathfrak{R}\delta\mathfrak{r} + \text{etc.}\\ - M\delta x - Np\delta x - Pq\delta x - Qr\delta x - Rs\delta x - \text{etc.}\\ - \mathfrak{N}\mathfrak{p}\delta x - \mathfrak{P}\mathfrak{q}\delta x - \mathfrak{Q}\mathfrak{r}\delta x - \mathfrak{R}\mathfrak{s}\delta x - \text{etc.}\end{aligned}$$

Quodsi iam brevitatis gratia statuamus

$$\delta y - p\delta x = \omega \quad\text{et}\quad \delta z - \mathfrak{p}\delta x = \mathfrak{w},$$

sumto elemento dx constante ex § 125 et 126 erit

$$\begin{aligned}&\delta p - q\delta x = \frac{d\omega}{dx}, && \delta\mathfrak{p} - \mathfrak{q}\delta x = \frac{d\mathfrak{w}}{dx},\\ &\delta q - r\delta x = \frac{dd\omega}{dx^2}, && \delta\mathfrak{q} - \mathfrak{r}\delta x = \frac{dd\mathfrak{w}}{dx^2},\\ &\delta r - s\delta x = \frac{d^3\omega}{dx^3}, && \delta\mathfrak{r} - \mathfrak{s}\delta x = \frac{d^3\mathfrak{w}}{dx^3}\end{aligned}$$

etc.,

unde variatio quaesita hoc modo commode exprimetur

$$\delta\int V dx = V\delta x + \int dx \left\{ \begin{array}{l} N\omega + \dfrac{P d\omega}{dx} + \dfrac{Q dd\omega}{dx^2} + \dfrac{R d^3\omega}{dx^3} + \text{etc.} \\ + \mathfrak{N}\mathfrak{w} + \dfrac{\mathfrak{P} d\mathfrak{w}}{dx} + \dfrac{\mathfrak{Q} dd\mathfrak{w}}{dx^2} + \dfrac{\mathfrak{R} d^3\mathfrak{w}}{dx^3} + \text{etc.} \end{array} \right\},$$

quae ut supra [§ 80—85] ad hanc formam reducitur

$$\begin{aligned} \delta\int V dx = & \int \omega dx \left(N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}\right) \\ & + \int \mathfrak{w} dx \left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\right) \\ & + \quad V\delta x + \omega \left(P - \frac{dQ}{dx} + \frac{ddR}{dx^2} - \frac{d^3S}{dx^3} + \text{etc.}\right) \\ & + \text{Const.} + \mathfrak{w} \left(\mathfrak{P} - \frac{d\mathfrak{Q}}{dx} + \frac{dd\mathfrak{R}}{dx^2} - \frac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right) \\ & \qquad + \frac{d\omega}{dx}\left(Q - \frac{dR}{dx} + \frac{ddS}{dx^2} - \text{etc.}\right) \\ & \qquad + \frac{d\mathfrak{w}}{dx}\left(\mathfrak{Q} - \frac{d\mathfrak{R}}{dx} + \frac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right) \\ & \qquad + \frac{dd\omega}{dx^2}\left(R - \frac{dS}{dx} + \text{etc.}\right) \\ & \qquad + \frac{dd\mathfrak{w}}{dx^2}\left(\mathfrak{R} - \frac{d\mathfrak{S}}{dx} + \text{etc.}\right) \\ & \qquad + \frac{d^3\omega}{dx^3}(S - \text{etc.}) \\ & \qquad + \frac{d^3\mathfrak{w}}{dx^3}(\mathfrak{S} - \text{etc.}) \\ & \qquad + \text{etc.}, \end{aligned}$$

cuius indoles ex superioribus satis est manifesta, eademque circa constantis additionem sunt observanda [§ 115].

COROLLARIUM 1

130. In hac solutione ambae variabiles y et z tanquam functiones ipsius x spectantur, sive iam sint cognitae sive demum ex variationis indole definiendae. Neque etiam formula integralis $\int V dx$ certum esset habitura valorem, nisi tam y quam z per x determinari conciperetur.

COROLLARIUM 2

131. Si formula Vdx per se sit integrabilis nulla assumta relatione inter ternas variabiles, variatio integralis $\int Vdx$ nullas quoque formulas integrales involvere potest ideoque necesse est, ut tum sit

$$\text{et } N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

$$\text{et } \mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.} = 0.$$

COROLLARIUM 3

132. Vicissim etiam si hae duae aequationes locum habeant, hoc certum erit criterium formulam differentialem Vdx per se integrationem admittere nulla inter variabiles stabilita relatione.

EXEMPLUM

133. *Quo hoc criterium magis illustremus, sumamus eiusmodi formulam per se integrabilem sitque* $\int Vdx = \frac{z\,dy}{x\,dz} = \frac{pz}{x\mathfrak{p}}$, *unde fit*

$$V = \frac{-pz}{xx\mathfrak{p}} + \frac{p}{x} + \frac{zq}{x\mathfrak{p}} - \frac{zp\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}}.$$

Ex cuius differentiatione colligimus

$$N = 0 \quad \text{et} \quad P = \frac{-z}{xx\mathfrak{p}} + \frac{1}{x} - \frac{z\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}}, \quad Q = \frac{z}{x\mathfrak{p}},$$

porro

$$\mathfrak{N} = \frac{-p}{xx\mathfrak{p}} + \frac{q}{x\mathfrak{p}} - \frac{p\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}}, \quad \mathfrak{P} = \frac{pz}{xx\mathfrak{p}\mathfrak{p}} - \frac{zq}{x\mathfrak{p}\mathfrak{p}} + \frac{2zp\mathfrak{q}}{x\mathfrak{p}^3} \quad \text{et} \quad \mathfrak{Q} = -\frac{zp}{x\mathfrak{p}\mathfrak{p}}.$$

Iam pro prima aequatione ob $N = 0$ fieri oportet

$$-\frac{dP}{dx} + \frac{ddQ}{dx^2} = 0 \quad \text{seu} \quad P - \frac{dQ}{dx} = \text{Const.},$$

cuius veritas ex differentiatione ipsius Q statim fit perspicua.

Pro altera aequatione

$$\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} = 0,$$

quia hinc est $\int \mathfrak{R}dx = \mathfrak{P} - \frac{d\mathfrak{Q}}{dx}$, primo necesse est, ut integrabilis existat haec formula

$$\mathfrak{R}dx = \frac{-pdx}{xx\mathfrak{p}} + \frac{qdx}{x\mathfrak{p}} - \frac{p\mathfrak{q}dx}{x\mathfrak{p}\mathfrak{p}},$$

unde ob $qdx = dp$ manifesto fit $\int \mathfrak{R}dx = \frac{p}{x\mathfrak{p}}$. Superest ergo, ut sit

$$\frac{d\mathfrak{Q}}{dx} = \mathfrak{P} - \int \mathfrak{R}dx = \frac{pz}{xx\mathfrak{p}\mathfrak{p}} - \frac{zq}{x\mathfrak{p}\mathfrak{p}} + \frac{2zp\mathfrak{q}}{x\mathfrak{p}^3} - \frac{p}{x\mathfrak{p}}.$$

Verum differentiando $\mathfrak{Q} = \frac{-zp}{x\mathfrak{p}\mathfrak{p}}$ utrinque perfecta aequalitas resultat.

SCHOLION 1

134. Quodsi ergo quaestio huc redeat, ut formulae integrali $\int Vdx$ valor maximus minimusve sit conciliandus, tum ante omnia in eius variatione ambas partes integrales idque seorsim nihilo aequari oportet, propterea quod, utcunque variationes constituantur, variatio $\delta \int Vdx$ semper debeat evanescere; unde duae emergunt aequationes istae

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

et

$$\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.} = 0,$$

quibus duplex relatio inter ternas variabiles x, y, z ita exprimitur, ut deinceps tam y quam z recte tanquam functio ipsius x spectari possit. Quando autem hae aequationes sunt differentiales idque altioris gradus, totidem utrinque constantes arbitrariae per integrationes in calculum invehuntur, quoti gradus utraque fuerit differentialis. Has vero constantes deinceps ita definiri oportet, ut conditionibus tam pro initio quam pro fine integrationis formulae $\int Vdx$ praescriptis satisfiat, quod negotium eo redit, ut praeterea variationis partes absolutae ad nihilum redigantur. Primo scilicet Constans ita definiri debet, ut conditionibus pro initio praescriptis satisfiat, ubi quidem ex quaestionis indole particulae

$$\omega, \quad \mathfrak{w}, \quad \frac{d\omega}{dx}, \quad \frac{d\mathfrak{w}}{dx}, \quad \frac{dd\omega}{dx^2}, \quad \frac{dd\mathfrak{w}}{dx^2} \quad \text{etc.}$$

definitos valores sortiri solent. Tum vero, cum idem circa finem integrationis usu veniat, ex singulis constantes per integrationem ingressae determinabuntur.

SCHOLION 2

135. Plurimum conducet hic observasse membra, quibus variatio $\delta\int V dx$ exprimitur, sponte in duas classes dispesci, in quarum altera litterae tantum eae conspiciuntur, quae ad variabilitatem ipsius y seu ad eius habitum respectu x referuntur idque ita, ac si quantitas z constans esset assumta, altera vero classis similes litteras a variabilitate ipsius z tantum pendentes continet, quasi quantitas y esset constans. Ex quo colligere licet, si etiam quarta variabilis v accedat, quae ut functio ipsius x quoque spectari queat, tum ad illas duas classes tertiam insuper esse adiiciendam, quae similia membra a variabilitate solius v pendentia complectatur. Quocirca solutio hic data spectari potest, quasi ad quotcunque variabiles extendatur, dummodo tot inter eas aequationes dari concipiantur, ut omnes pro functionibus unius haberi queant. Etsi ergo hoc caput tantum tres variabiles prae se fert, tamen ad quotcunque pertinere est intelligendum, si modo eiusmodi conditiones proponantur, ut tandem per unam reliquae omnes determinentur. Talem autem conditionem formulae integrales huius formae $\int V dx$ necessario involvunt; quotcunque enim variabiles in quantitatem V ingrediantur, expressio $\int V dx$ certum valorem definitum omnino obtinere nequit, nisi omnes variabiles tanquam functiones unius x spectari queant. Longe aliter autem est comparata ratio earum formularum integralium, quae ad duas pluresve variabiles a se invicem minime pendentes referuntur.

PROBLEMA 14

136. *Si functio V praeter tres variabiles x, y, z earumque differentialia cuiuscunque gradus insuper involvat formulam integralem $v = \int \mathfrak{V} dx$, ubi $\mathfrak{V}$ sit functio quaecunque earundem variabilium x, y, z cum suis differentialibus, investigare variationem formulae integralis $\int V dx$.*

SOLUTIO

Ut species saltem differentialium e calculo tollatur, ponamus ut ante

$$dy = p dx, \quad dp = q dx, \quad dq = r dx, \quad dr = s dx \quad \text{etc.},$$

$$dz = \mathfrak{p} dx, \quad d\mathfrak{p} = \mathfrak{q} dx, \quad d\mathfrak{q} = \mathfrak{r} dx, \quad d\mathfrak{r} = \mathfrak{s} dx \quad \text{etc.}$$

ac functione V differentiata prodeat

$$\begin{aligned} dV = Ldv + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.} \\ + \mathfrak{N}dz + \mathfrak{P}d\mathfrak{p} + \mathfrak{Q}d\mathfrak{q} + \mathfrak{R}d\mathfrak{r} + \text{etc.}; \end{aligned}$$

tum vero ob $dv = \mathfrak{V}dx$ sit

$$\begin{aligned} d\mathfrak{V} = M'dx + N'dy + P'dp + Q'dq + R'dr + \text{etc.} \\ + \mathfrak{N}'dz + \mathfrak{P}'d\mathfrak{p} + \mathfrak{Q}'d\mathfrak{q} + \mathfrak{R}'d\mathfrak{r} + \text{etc.}, \end{aligned}$$

ubi ob defectum litterarum iisdem accentu distinctis utor. Hinc autem simul earundem quantitatum V et $\mathfrak{V}$ variationes habentur. Iam cum quaeratur variatio $\delta\int Vdx$, habebimus primo quidem ut ante

$$\delta\int Vdx = V\delta x + \int(dx\delta V - dV\delta x);$$

ubi cum valor ipsius V non discrepet a praecedente, nisi quod hic ad eius differentiale dV accedat pars $Ldv = L\mathfrak{V}dx$ et ad variationem δV haec pars $L\delta v = L\delta\int\mathfrak{V}dx$, etiam variatio quaesita $\delta\int Vdx$ forma ante inventa exprimetur, si modo ad eam adiiciatur hoc membrum

$$\int L(dx\delta\int\mathfrak{V}dx - \mathfrak{V}dx\delta x) = \int Ldx(\delta\int\mathfrak{V}dx - \mathfrak{V}\delta x).$$

Quia vero formula integralis $\int\mathfrak{V}dx$ eadem est, quae in problemate praecedente est tractata, si, ut ibi fecimus, statuamus

$$\delta y - p\delta x = \omega \quad \text{et} \quad \delta z - \mathfrak{p}\delta x = \mathfrak{w},$$

elemento dx constante assumto habebimus

$$\delta\int\mathfrak{V}dx - \mathfrak{V}\delta x = \int dx \left\{ \begin{aligned} N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.} \\ + \mathfrak{N}'\mathfrak{w} + \frac{\mathfrak{P}'d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}'dd\mathfrak{w}}{dx^2} + \frac{\mathfrak{R}'d^3\mathfrak{w}}{dx^3} + \text{etc.} \end{aligned} \right\}.$$

Ponamus iam integrale $\int Ldx = I$, si scilicet ita capiatur, ut pro initio integrationis evanescat, tum vero pro termino finali integrationis fiat $I = A$, quo facto pro tota integrationis extensione erit

$$\int Ldx(\delta\int\mathfrak{V}dx - \mathfrak{V}\delta x) = \int(A - I)dx \left\{ \begin{aligned} N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \text{etc.} \\ + \mathfrak{N}'\mathfrak{w} + \frac{\mathfrak{P}'d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}'dd\mathfrak{w}}{dx^2} + \text{etc.} \end{aligned} \right\}.$$

Nunc igitur introducamus sequentes abbreviationes

$$\begin{array}{ll} N+(A-I)N'=N^0, & \mathfrak{N}+(A-I)\mathfrak{N}'=\mathfrak{N}^0, \\ P+(A-I)P'=P^0, & \mathfrak{P}+(A-I)\mathfrak{P}'=\mathfrak{P}^0, \\ Q+(A-I)Q'=Q^0, & \mathfrak{Q}+(A-I)\mathfrak{Q}'=\mathfrak{Q}^0, \\ R+(A-I)R'=R^0 & \mathfrak{R}+(A-I)\mathfrak{R}'=\mathfrak{R}^0 \\ \text{etc.} & \text{etc.} \end{array}$$

atque manifestum est variationem quaesitam ita expressam iri

$$\delta\int Vdx = V\delta x + \int dx\left\{\begin{array}{l} N^0\omega + \dfrac{P^0 d\omega}{dx} + \dfrac{Q^0 dd\omega}{dx^2} + \dfrac{R^0 d^3\omega}{dx^3} + \text{etc.} \\ + \mathfrak{N}^0\mathfrak{w} + \dfrac{\mathfrak{P}^0 d\mathfrak{w}}{dx} + \dfrac{\mathfrak{Q}^0 dd\mathfrak{w}}{dx^2} + \dfrac{\mathfrak{R}^0 d^3\mathfrak{w}}{dx^3} + \text{etc.} \end{array}\right\},$$

quae etiam ut ante evolvitur in hanc formam

$$\begin{aligned} \delta\int Vdx = {} & \int\omega dx\left(N^0 - \frac{dP^0}{dx} + \frac{ddQ^0}{dx^2} - \frac{d^3R^0}{dx^3} + \frac{d^4S^0}{dx^4} - \text{etc.}\right) \\ & + \int\mathfrak{w}\,dx\left(\mathfrak{N}^0 - \frac{d\mathfrak{P}^0}{dx} + \frac{dd\mathfrak{Q}^0}{dx^2} - \frac{d^3\mathfrak{R}^0}{dx^3} + \frac{d^4\mathfrak{S}^0}{dx^4} - \text{etc.}\right) \\ + V\delta x + {} & \omega\left(P^0 - \frac{dQ^0}{dx} + \frac{ddR^0}{dx^2} - \frac{d^3S^0}{dx^3} + \text{etc.}\right) \\ + \text{Const.} + {} & \mathfrak{w}\left(\mathfrak{P}^0 - \frac{d\mathfrak{Q}^0}{dx} + \frac{dd\mathfrak{R}^0}{dx^2} - \frac{d^3\mathfrak{S}^0}{dx^3} + \text{etc.}\right) \\ & + \frac{d\omega}{dx}\left(Q^0 - \frac{dR^0}{dx} + \frac{ddS^0}{dx^2} - \text{etc.}\right) \\ & + \frac{d\mathfrak{w}}{dx}\left(\mathfrak{Q}^0 - \frac{d\mathfrak{R}^0}{dx} + \frac{dd\mathfrak{S}^0}{dx^2} - \text{etc.}\right) \\ & + \frac{dd\omega}{dx^2}\left(R^0 - \frac{dS^0}{dx} + \text{etc.}\right) \\ & + \frac{dd\mathfrak{w}}{dx^2}\left(\mathfrak{R}^0 - \frac{d\mathfrak{S}^0}{dx} + \text{etc.}\right) \\ & + \frac{d^3\omega}{dx^3}\left(S^0 - \text{etc.}\right) \\ & + \frac{d^3\mathfrak{w}}{dx^3}\left(\mathfrak{S}^0 - \text{etc.}\right) \\ & + \text{etc.}, \end{aligned}$$

ubi neminem offendat signum nihili litteris suffixum, siquidem non exponentem denotat, sed tantum ad has litteras ab iisdem nude positis distinguendas adhibetur.

COROLLARIUM 1

137. Si igitur formula integralis $\int V dx$ habere debeat valorem maximum vel minimum, variationis inventae bina membra priora statim nihilo aequalia statui oportet, unde duae resultant aequationes differentiales, quibus indefinita relatio utriusque variabilis y et z ad x definitur.

COROLLARIUM 2

138. Etiamsi hic conditionum, quae forte pro initio et fine integrationis proponantur, nondum ratio habetur, tamen eae iam occulte in calculum ingrediuntur, quia litterae I et A terminos integrationis respiciunt. Interim tamen eae in ipsa aequationum differentialium tractatione iterum ex calculo expelluntur; dum enim formula integralis $\int L dx = I$ eliditur, simul quantitas constans A egreditur.

COROLLARIUM 3

139. Expeditis autem aequationibus his duabus differentialibus idque generalissime, ut totidem constantes arbitrariae in calculum invehantur, quot integrationes institui oportuit, tum demum ad conditiones utriusque termini integrationis formulae $\int V dx$ est attendendum, quandoquidem hinc ex reliquis variationis membris absolutis illae constantes determinari debent.

SCHOLION

140. Solutio huius problematis ita est comparata, ut iam satis sit perspicuum, quemadmodum etiam formulas magis complicatas, veluti si functio V plures formulas integrales involvat vel si quoque functio $\mathfrak{V}$ formulas novas integrales complectatur, expediri conveniat. Quin etiamnunc est manifestum, si huiusmodi formulae integrales plures tribus variabiles contineant, quomodo tum variationes inveniri oporteat, atque adeo non solum taediosum, sed etiam superfluum foret, si copiosius hoc argumentum persequi vellem. Ad partem igitur huius doctrinae alteram multo abstrusiorem progredior, ubi etiam relationibus inter variabiles constitutis duae pluresve a se invicem minime pendentes in calculo relinquuntur.

CAPUT VI

DE VARIATIONE FORMULARUM DIFFERENTIALIUM TRES VARIABILES INVOLVENTIUM QUARUM RELATIO UNICA AEQUATIONE CONTINETUR

PROBLEMA 15

141. *Proposita aequatione inter tres variabiles x, y et z, quibus variationes quaecunque δx, δy, δz tribuuntur, definire variationes formularum differentialium primi gradus*

$$p = \left(\frac{dz}{dx}\right) \quad et \quad p' = \left(\frac{dz}{dy}\right).$$

SOLUTIO

Cum unica aequatio inter tres variabiles dari ponitur, quaelibet earum tanquam functio binarum reliquarum spectari potest. Erit ergo z functio ipsarum x et y et meminisse hic oportet expressionem $\left(\frac{dz}{dx}\right) = p$ denotare rationem differentialium ipsarum z et x, si in aequatione illa data hae solae ut variabiles tractentur tertia y pro constante habita, quod idem de altera formula $p' = \left(\frac{dz}{dy}\right)$ est tenendum. Simili modo ipsae quoque variationes δx, δy, δz ut functiones infinite parvae binarum variabilium x et y spectari possunt, quoniam, si etiam a tertia z penderent, haec ipsa est functio ipsarum x et y; unde simul intelligitur, quid istae formulae

$$\left(\frac{d\delta z}{dx}\right), \quad \left(\frac{d\delta z}{dy}\right), \quad \text{item} \quad \left(\frac{d\delta x}{dx}\right), \quad \left(\frac{d\delta x}{dy}\right) \quad \text{et} \quad \left(\frac{d\delta y}{dx}\right), \quad \left(\frac{d\delta y}{dy}\right)$$

significent. Cum igitur valor variatus formulae $\left(\frac{dz}{dx}\right)=p$ sit

$$p+\delta p=\left(\frac{d(z+\delta z)}{d(x+\delta x)}\right),$$

si scilicet hic variabilis y constans sumatur, erit hac conditione observata

$$p+\delta p=\left(\frac{dz+d\delta z}{dx+d\delta x}\right)=\left(\frac{dz}{dx}+\frac{d\delta z}{dx}-\frac{dz\,d\delta x}{dx^2}\right),$$

propterea quod variationes δdx et δdz prae dx et dz evanescunt.[1]) Hinc ergo ob $\left(\frac{dz}{dx}\right)=p$ habebitur variatio quaesita[2])

$$\delta p=\left(\frac{d\delta z}{dx}\right)-\left(\frac{dz}{dx}\cdot\frac{d\delta x}{dx}\right)=\left(\frac{d\delta z}{dx}\right)-p\left(\frac{d\delta x}{dx}\right),$$

quarum formularum significatus, cum tam δz quam δx sint functiones ipsarum x et y hicque y constans habeatur, per se est manifestus. Simili autem modo reperietur fore

$$\delta p'=\left(\frac{d\delta z}{dy}\right)-p'\left(\frac{d\delta y}{dy}\right),$$

ubi iam variabilis x pro constante habetur.

COROLLARIUM 1

142. Hic omnia ad binas variabiles x et y sunt perducta atque ut earum functiones spectantur non solum tertia z, sed etiam omnes tres variationes δx, δy, δz; manifestum autem est has tres variabiles pro lubitu inter se permutari posse.

COROLLARIUM 2

143. Sufficit autem his binis formulis pro differentialibus primi gradus uti, quoniam reliquas ad has reducere licet, siquidem sit

$$\left(\frac{dx}{dz}\right)=\frac{1}{p},\quad \left(\frac{dy}{dz}\right)=\frac{1}{p'}\quad \text{et}\quad \left(\frac{dy}{dx}\right)=\frac{-p}{p'}\quad \text{et}\quad \left(\frac{dx}{dy}\right)=\frac{-p'}{p},$$

ubi p et p' sunt functiones binarum x et y.

---

1) Editio princeps: *propterea quod variationes δx et δz prae x et z evanescunt.* Correxit F. E.

2) Facile intelligitur hanc formulam non subsistere, nisi δy a variabili y sola pendeat, quod quidem auctor ipse in § 148 observat discrepantia duorum valorum pro $\delta q'$ in § 147 inventorum monitus. F. E.

COROLLARIUM 3

144. Inventis ergo variationibus harum duarum formularum

$$p = \left(\frac{dz}{dx}\right) \quad \text{et} \quad p' = \left(\frac{dz}{dy}\right)$$

reliquarum formularum modo memoratarum variationes hinc facile reperientur. Erit enim

$$\delta\left(\frac{dx}{dz}\right) = -\frac{\delta p}{pp} = -\frac{1}{pp}\left(\frac{d\delta z}{dx}\right) + \frac{1}{p}\left(\frac{d\delta x}{dx}\right),$$

$$\delta\left(\frac{dy}{dz}\right) = -\frac{\delta p'}{p'p'} = -\frac{1}{p'p'}\left(\frac{d\delta z}{dy}\right) + \frac{1}{p'}\left(\frac{d\delta y}{dy}\right),$$

$$\delta\left(\frac{dy}{dx}\right) = -\frac{\delta p}{p'} + \frac{p\,\delta p'}{p'p'} = -\frac{1}{p'}\left(\frac{d\delta z}{dx}\right) + \frac{p}{p'}\left(\frac{d\delta x}{dx}\right) + \frac{p}{p'p'}\left(\frac{d\delta z}{dy}\right) - \frac{p}{p'}\left(\frac{d\delta y}{dy}\right).$$

SCHOLION 1

145. Hic ante omnia observo formulas differentiales certum valorem habere non posse, nisi duo differentialia ita inter se comparentur, ut tertia variabilis, si tres habeantur, seu reliquae omnes, si plures adsint, constantes accipiantur. Ita hoc casu, quo inter tres variabiles x, y et z unica aequatio datur vel saltem dari concipitur, formula $\frac{dz}{dx}$ nullum plane habet significatum, nisi tertia variabilis y constans sumatur, quam conditionem vinculis includendo hanc formulam innuere consueverunt, etiamsi ea tuto omitti possent, quoniam alioquin ne ullus quidem significatus adesset. Quod quo magis perspicuum reddatur, quaecunque aequatio inter ternas variabiles x, y, z proponatur, ex ea valor ipsius z elici concipiatur, ut z aequetur certae functioni ipsarum x et y eiusque sumto differentiali prodeat $dz = pdx + p'dy$, ubi iterum p et p' certae erunt functiones ipsarum x et y idque tales, ut sit $\left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right)$. Sumta nunc y constante fit $dz = pdx$ seu $p = \left(\frac{dz}{dx}\right)$, sumta autem x constante prodit $p' = \left(\frac{dz}{dy}\right)$. Tum vero etiam manifestum est sumta z constante fore $\frac{dy}{dx} = \frac{-p}{p'}$; huiusmodi autem formulas excludi conveniet, quando tam z quam variationes δx, δy et δz ut functiones ipsarum x et y repraesentamus.

SCHOLION 2

146. Ex Geometria hoc argumentum multo clarius illustrare licet. Denotent enim tres nostrae variabiles x, y, z ternas coordinatas AX, XY, YZ

(Fig. 4, p. 387), inter quas aequatio proposita certam quandam superficiem assignabit, in qua ordinata $YZ = z$ terminabitur, quae utique tanquam certa functio binarum reliquarum $AX = x$ et $XY = y$ spectari potest, ita ut sumtis pro lubitu his binis x et y tertia $YZ = z$ ex aequatione proposita determinetur. Quodsi iam alia superficies quaecunque concipiatur ab ista infinite parum discrepans eaque ita cum hac comparetur, ut eius punctum quodvis z cum propositae puncto Z conferatur, ita tamen, ut intervallum Zz sit semper infinite parvum, variationes ita repraesentabuntur, ut sit

$$\delta x = Ax - AX = Xx, \quad \delta y = xy - XY \quad \text{et} \quad \delta z = yz - YZ;$$

et cum hae variationes prorsus arbitrio nostro permittantur neque ullo modo a se invicem pendeant, eae etiam tanquam functiones binarum x et y spectari possunt idque ita, ut nulla a reliquis pendeat, sed unaquaeque pro arbitrio fingi queat. Quin etiam hinc intelligitur, quoniam superficies proxima a proposita diversa esse debet, neutiquam fore

$$\delta z = p\delta x + p'\delta y,$$

siquidem pro superficie proposita fuerit

$$dz = pdx + p'dy;$$

alioquin punctum z foret in eadem superficie, ex quo omnino ternas functiones ipsarum x et y pro variationibus δx, δy et δz ita comparatas esse oportet, ut non sit

$$\delta z = p\delta x + p'\delta y,$$

sed potius ab hoc valore quomodocunque discrepet; ubi quidem imprimis notandum est has functiones ita late patere, ut discontinuae non excludantur atque adeo pro lubitu variationes tantum in unico puncto vel saltem exiguo spatio constitui queant. Ne autem hic ulli dubio locus relinquatur, probe notandum est ex eo, quod ponimus z eiusmodi functionem ipsarum x et y, ut sit

$$dz = pdx + p'dy,$$

minime sequi fore quoque

$$\delta z = p\delta x + p'\delta y,$$

quemadmodum supra assumsimus, propterea quod hic ipsi z propriam tribuimus variationem neutiquam pendentem a variationibus ipsarum x et y.

PROBLEMA 16

147. *Proposita aequatione inter tres variabiles x, y, z, quibus variationes quaecunque δx, δy, δz tribuuntur, investigare variationes formularum differentialium secundi gradus*

$$q = \left(\frac{ddz}{dx^2}\right), \quad q' = \left(\frac{ddz}{dx\,dy}\right) \quad et \quad q'' = \left(\frac{ddz}{dy^2}\right)$$

SOLUTIO

Hic iterum z spectatur ut functio ipsarum x et y, quarum etiam sunt functiones ternae variationes δx, δy, δz nullo modo a se invicem pendentes. Quoniam in praecedente problemate posuimus

$$p = \left(\frac{dz}{dx}\right) \quad \text{et} \quad p' = \left(\frac{dz}{dy}\right),$$

his formulis in subsidium vocatis habebimus

$$q = \left(\frac{dp}{dx}\right), \quad q' = \left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right) \quad \text{et} \quad q'' = \left(\frac{dp'}{dy}\right)$$

hicque ratio variationum δp et $\delta p'$ est habenda, quas invenimus

$$\delta p = \left(\frac{d\delta z}{dx}\right) - p\left(\frac{d\delta x}{dx}\right) \quad \text{et} \quad \delta p' = \left(\frac{d\delta z}{dy}\right) - p'\left(\frac{d\delta y}{dy}\right).$$

Simili ergo modo calculum subducendo reperiemus primo

$$\delta q = \left(\frac{d\delta p}{dx}\right) - q\left(\frac{d\delta x}{dx}\right),$$

ubi $\left(\frac{d\delta p}{dx}\right)$ invenitur, si valor δp differentietur posita y constante ac differentiale per dx dividatur, unde oritur

$$\left(\frac{d\delta p}{dx}\right) = \left(\frac{dd\delta z}{dx^2}\right) - q\left(\frac{d\delta x}{dx}\right) - p\left(\frac{dd\delta x}{dx^2}\right)$$

ob $q = \left(\frac{dp}{dx}\right)$, unde concludimus

$$\delta q = \left(\frac{dd\delta z}{dx^2}\right) - 2q\left(\frac{d\delta x}{dx}\right) - p\left(\frac{dd\delta x}{dx^2}\right).$$

Eodem modo ob $q' = \left(\frac{dp}{dy}\right)$ erit

$$\delta q' = \left(\frac{d\delta p}{dy}\right) - q'\left(\frac{d\delta y}{dy}\right),$$

at

$$\left(\frac{d\delta p}{dy}\right) = \left(\frac{dd\delta z}{dx\,dy}\right) - q'\left(\frac{d\delta x}{dx}\right) - p\left(\frac{dd\delta x}{dx\,dy}\right)$$

ideoque

$$\delta q' = \left(\frac{dd\delta z}{dx\,dy}\right) - q'\left(\frac{d\delta x}{dx}\right) - q'\left(\frac{d\delta y}{dy}\right) - p\left(\frac{dd\delta x}{dx\,dy}\right).$$

Alter autem valor $q' = \left(\frac{dp'}{dx}\right)$ simili modo tractatus praebet

$$\delta q' = \left(\frac{dd\delta z}{dx\,dy}\right) - q'\left(\frac{d\delta x}{dx}\right) - q'\left(\frac{d\delta y}{dy}\right) - p'\left(\frac{dd\delta y}{dx\,dy}\right),$$

cuius valoris ab illo discrepantia incommodum involvit mox accuratius examinandum. Ex tertia autem formula $q'' = \left(\frac{dp'}{dy}\right)$ elicitur

$$\delta q'' = \left(\frac{dd\delta z}{dy^2}\right) - 2q''\left(\frac{d\delta y}{dy}\right) - p'\left(\frac{dd\delta y}{dy^2}\right).$$

SCHOLION 1

148. In originem discrepantiae variationis $\delta q'$ ex gemino valore

$$q' = \left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right)$$

natae inquisiturus observo in his formulis variationem exprimentibus vel quantitatem x vel quantitatem y pro constanti haberi, prout denominator cuiuscunque membri declarat. Verum si quantitatem x constantem manere sumimus, utcunque interea altera y mutabilis existit, natura rei postulat, ut etiam variationes ipsius x nullam mutationem subeant, quod autem secus evenit, si variatio δx quoque a quantitate y pendeat, quod idem de altera variabili y, dum constans ponitur, est tenendum. Ex quo manifestum est, si variationes δx et δy simul ab ambabus variabilibus x et y pendere sumantur, id ipsi hypothesi, qua alterutra perpetuo constans ponitur, adversari. Quamobrem hoc incommodum aliter vitari nequit, nisi statuamus variationem ipsius x prorsus non ab altera variabili y neque huius variationem δy ab

altera x pendere. Sin autem δx per solam x et δy per solam y determinatur, ut sit

$$\text{et}\quad \left(\frac{d\delta x}{dy}\right)=0 \quad\text{et}\quad \left(\frac{d\delta y}{dx}\right)=0,$$

erit etiam

$$\left(\frac{dd\delta x}{dx\,dy}\right)=0 \quad\text{et}\quad \left(\frac{dd\delta y}{dx\,dy}\right)=0$$

sicque ambo illi valores discrepantes pro $\delta q'$ inventi ad consensum perducuntur.

SCHOLION 2

149. Omnibus autem dubiis in hac investigatione felicissime occurremus, si soli quantitati z variationes tribuamus binis reliquis x et y plane invariatis relictis, ita ut sit tam $\delta x=0$ quam $\delta y=0$, quo pacto non solum calculo consulitur, sed etiam usus huius calculi variationum vix restringitur. Quodsi enim superficiem quamcunque cum alia sibi proxima comparamus, nihil impedit, quominus singula proposita superficiei puncta ad ea proximae puncta referamus, quibus eaedem binae coordinatae x et y respondeant, solaque tertia z variationem patiatur. Quin etiam haec suppositio, cum ad formulas integrales progrediemur, eo magis est necessaria, quandoquidem semper totum negotium ad eiusmodi formulas integrales perducitur, quae duplicem integrationem requirunt, in quarum altera sola x, in altera vero sola y ut variabilis tractatur; nisi ergo harum variationes nullae statuantur, maxima incommoda inde in calculum inveherentur; qui cum per se plerumque sit difficillimus, minime consultum videtur, ut ex hac parte difficultates multiplicentur. Quamobrem hanc tractationem ita sum expediturus, ut in posterum perpetuo binis variabilibus x et y nullas plane variationes tribuam solamque tertiam z variatione quacunque δz augeri assumam, ubi quidem δz ut functionem quamcunque ipsarum x et y sive continuam sive discontinuam sum spectaturus.

PROBLEMA 17

150. *Si z fuerit functio quaecunque ipsarum x et y eique tribuatur variatio δz pariter utcunque ab x et y pendens, investigare variationes formularum omnium differentialium cuiuscunque ordinis.*

SOLUTIO

Pro differentialibus primi gradus habentur hae duae formulae

$$p = \left(\frac{dz}{dx}\right) \quad \text{et} \quad p' = \left(\frac{dz}{dy}\right),$$

quarum variationes, cum x et y nullam variationem pati concipiantur, ex supra inventis ita [se] habebunt

$$\delta p = \left(\frac{d\delta z}{dx}\right) \quad \text{et} \quad \delta p' = \left(\frac{d\delta z}{dy}\right).$$

Pro differentialibus secundi ordinis hae tres formulae habentur

$$q = \left(\frac{ddz}{dx^2}\right), \quad q' = \left(\frac{ddz}{dx\,dy}\right) \quad \text{et} \quad q'' = \left(\frac{ddz}{dy^2}\right),$$

ita ut sit

$$q = \left(\frac{dp}{dx}\right), \quad q' = \left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right) \quad \text{et} \quad q'' = \left(\frac{dp'}{dy}\right),$$

quarum variationes ex praecedente problemate ob $\delta x = 0$ et $\delta y = 0$ sunt

$$\delta q = \left(\frac{dd\delta z}{dx^2}\right), \quad \delta q' = \left(\frac{dd\delta z}{dx\,dy}\right), \quad \delta q'' = \left(\frac{dd\delta z}{dy^2}\right).$$

Simili modo si ad differentialia tertii ordinis ascendamus, hae quatuor formulae occurrunt

$$r = \left(\frac{d^3 z}{dx^3}\right), \quad r' = \left(\frac{d^3 z}{dx^2\,dy}\right), \quad r'' = \left(\frac{d^3 z}{dx\,dy^2}\right), \quad r''' = \left(\frac{d^3 z}{dy^3}\right),$$

quarum variationes ita expressum iri manifestum est

$$\delta r = \left(\frac{d^3 \delta z}{dx^3}\right), \quad \delta r' = \left(\frac{d^3 \delta z}{dx^2\,dy}\right), \quad \delta r'' = \left(\frac{d^3 \delta z}{dx\,dy^2}\right), \quad \delta r''' = \left(\frac{d^3 \delta z}{dy^3}\right),$$

unde per se patet, quomodo variationes formularum differentialium superiorum ordinum sint exprimendae.

COROLLARIUM 1

151. Hinc iam manifestum est fore in genere pro formula differentiali cuiuscunque ordinis $\left(\frac{d^{\mu+\nu} z}{dx^\mu\,dy^\nu}\right)$ eius variationem $= \left(\frac{d^{\mu+\nu} \delta z}{dx^\mu\,dy^\nu}\right)$, in qua forma superiores omnes continentur.

COROLLARIUM 2

152. Deinde etiam perspicuum est introducendis loco differentialium primi ordinis litteris p, p', secundi ordinis litteris q, q', q'', tertii ordinis litteris r, r', r'', r''', quarti ordinis litteris s, s', s'', s''', s^{IV} etc. speciem differentialium tolli, quemadmodum etiam supra huiusmodi litteris speciem differentialium sustulimus.

SCHOLION

153. Quoniam binae variabiles x et y prorsus a se invicem non pendent, ita ut altera adeo eundem valorem retinere queat, dum altera per omnes valores possibiles variatur, evidens huiusmodi formulam differentialem $\frac{dy}{dx}$, quippe quae nullum plane significatum certum esset habitura, in calculo nunquam locum invenire posse. Contra vero, cum quantitas z sit functio ipsarum x et y, hae formulae $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ et reliquae omnes, quas supra sum contemplatus, definitos habent significatus neque ullae aliae in calculum ingredi possunt. Deinde quia semper quaestiones huc pertinentes eo reducere licet, ut z tanquam functio binarum x et y spectari possit, eiusmodi formulae $\left(\frac{dy}{dx}\right)$, ubi quantitas z esset pro constanti habita, hinc prorsus excluduntur neque ullae aliae praeter supra memoratas in calculo admitti sunt censendae; sicque omnes expressiones a formulis integralibus liberae praeter ipsas variabiles x, y, z alias formulas differentiales non implicabunt praeter eas, quarum variationes hic sunt indicatae.

PROBLEMA 18

154. *Si z sit functio ipsarum x et y eique tribuatur variatio δz utcunque ab x et y pendens, tum vero fuerit V quantitas quomodocunque ex tribus variabilibus x, y, z earumque differentialibus cuiuscunque ordinis composita, eius variationem δV investigare.*

SOLUTIO

Ut in expressione V species differentialium tollantur, ponamus, ut hactenus fecimus,

$$p = \left(\frac{dz}{dx}\right), \quad p' = \left(\frac{dz}{dy}\right),$$

$$q = \left(\frac{ddz}{dx^2}\right), \quad q' = \left(\frac{ddz}{dx\,dy}\right), \quad q'' = \left(\frac{ddz}{dy^2}\right),$$

$$r = \left(\frac{d^3z}{dx^3}\right), \quad r' = \left(\frac{d^3z}{dx^2dy}\right), \quad r'' = \left(\frac{d^3z}{dx\,dy^2}\right), \quad r''' = \left(\frac{d^3z}{dy^3}\right)$$

etc.,

quarum formularum variationes a variatione ipsius z oriundas ita definimus, ut posita evidentiae gratia ista variatione $\delta z = \omega$, quam ut functionem quamcunque binarum variabilium x et y spectari oportet, sit

$$\delta p = \left(\frac{d\omega}{dx}\right), \quad \delta p' = \left(\frac{d\omega}{dy}\right),$$

$$\delta q = \left(\frac{dd\omega}{dx^2}\right), \quad \delta q' = \left(\frac{dd\omega}{dx\,dy}\right), \quad \delta q'' = \left(\frac{dd\omega}{dy^2}\right),$$

$$\delta r = \left(\frac{d^3\omega}{dx^3}\right), \quad \delta r' = \left(\frac{d^3\omega}{dx^2dy}\right), \quad \delta r'' = \left(\frac{d^3\omega}{dx\,dy^2}\right), \quad \delta r''' = \left(\frac{d^3\omega}{dy^3}\right)$$

etc.

Illis autem factis substitutionibus expressio proposita V fiet functio harum quantitatum $x, y, z, p, p', q, q', q'', r, r', r'', r'''$ etc. Eius ergo differentiale talem induet formam

$$\begin{aligned} dV = Ldx + Mdy + Ndz &+ Pdp + Qdq + Rdr \\ &+ P'dp' + Q'dq' + R'dr' \\ &+ Q''dq'' + R''dr'' \\ &+ R'''dr''' \end{aligned}$$

etc.

Quoniam nunc formula V eatenus tantum variationem recipit, quatenus quantitates, ex quibus componitur, variantur, binae autem x et y immunes statuuntur, eius variatio, quam quaerimus, erit

$$\begin{aligned} \delta V = N\delta z &+ P\delta p + Q\delta q + R\delta r \\ &+ P'\delta p' + Q'\delta q' + R'dr' \\ &+ Q''\delta q'' + R''\delta r'' \\ &+ R'''\delta r''' \end{aligned}$$

etc.,

ac si loco variationis δz scribamus ω, habebimus variationes inventas substituendo

$$\begin{aligned}\delta V = N\omega &+ P\left(\frac{d\omega}{dx}\right) + Q\left(\frac{dd\omega}{dx^2}\right) + R\left(\frac{d^3\omega}{dx^3}\right) \\ &+ P'\left(\frac{d\omega}{dy}\right) + Q'\left(\frac{dd\omega}{dx\,dy}\right) + R'\left(\frac{d^3\omega}{dx^2 dy}\right) \\ &+ Q''\left(\frac{dd\omega}{dy^2}\right) + R''\left(\frac{d^3\omega}{dx\,dy^2}\right) \\ &+ R'''\left(\frac{d^3\omega}{dy^3}\right) \\ &\text{etc.,}\end{aligned}$$

cuius formatio, si forte etiam differentialia altiorum graduum ingrediantur, per se est manifesta.

COROLLARIUM 1

155. Cum ω spectetur ut functio binarum variabilium x et y, singularum partium, quae variationem δV constituunt, significatus est determinatus atque haec variatio perfecte definita est censenda.

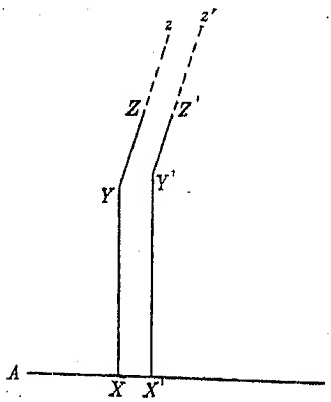

Fig. 6.

COROLLARIUM 2

156. Quomodocunque autem expressio V differentialibus sit referta, quandoquidem valorem certum indicare est censenda, substitutionibus adhibitis semper a specie differentialium liberari debet.

COROLLARIUM 3

157. Si nostrae tres variabiles ad superficiem referantur, ut sint eius coordinatae $AX = x$, $XY = y$, $YZ = z$ (Fig. 6), sola ordinata $YZ = z$ ubique incrementum infinite parvum $Zz = \delta z = \omega$ accipere intelligitur, ita ut puncta z cadant in aliam superficiem ab illa infinite parum discrepantem.

SCHOLION

158. Dubio hic occurri debet inde oriundo, quod quantitatem z ut functionem binarum x et y spectandam esse diximus; quoniam enim ipsis x et y

nullas variationes tribuimus, si in expressione V loco z eius valor in x et y substitueretur, ea ipsa in meram functionem ipsarum x et y abiret neque propterea ullam variationem esset receptura. Verum notandum est, tametsi z ut functio ipsarum x et y consideratur, eam tamen plerumque esse incognitam, quando scilicet eius naturam demum ex conditione variationis erui oportet; sin autem iam ab initio esset data, tamen, dum variatio quaeritur, functionem hanc z quasi incognitam spectari convenit minimeque eius loco valorem per x et y expressum substitui licet, antequam variatio, quippe quae a sola z pendet, penitus fuerit explorata.

CAPUT VII

DE VARIATIONE FORMULARUM INTEGRALIUM TRES VARIABILES INVOLVENTIUM QUARUM UNA UT FUNCTIO BINARUM RELIQUARUM SPECTATUR

PROBLEMA 19

159. *Formularum integralium huc pertinentium naturam evolvere ac rationem, qua earum variationes investigari conveniat, explicare.*

SOLUTIO

Cum tres habeantur variabiles x, y et z, quarum una z ut functio binarum reliquarum x et y est spectanda, etiamsi in ipsa variationis investigatione ratio huius functionis pro incognita haberi debet, formulae integrales, quae in hoc calculi genere occurrunt, plurimum discrepant ab iis, quae circa binas tantum variabiles proponi solent. Quemadmodum enim talis forma integralis $\int V dx$, ubi V duas variabiles x et y implicare censetur, quarum y ab x pendere concipitur, quasi summa omnium valorum elementarium $V dx$ per omnes valores ipsius x collectorum considerari potest, ita, quando tres variabiles x, y et z habentur, quarum haec z a binis x et y simul pendere concipitur, integralia huc pertinentia collectionem omnium elementorum ad omnes valores tam ipsius x quam ipsius y relatorum involvunt ideoque duplicem integrationem, alteram per omnes valores ipsius x, alteram vero ipsius y elementa congregantem requirunt. Ex quo huiusmodi integralia tali forma $\iint V dx dy$ contineri debent, qua scilicet duplex integratio innuatur, cuius

evolutio ita institui solet, ut primo altera variabilis y ut constans spectetur et formulae $\int V dx$ valor per terminos integrationis extensus quaeratur; in quo cum iam x obtineat valorem vel datum vel ab y pendentem, hoc integrale $\int V dx$ in functionem ipsius y tantum abibit, qua in dy ducta superest, ut integrale $\int dy \int V dx$ investigetur; quae ergo forma $\int dy \int V dx$ hoc modo tractata illi $\iint V dx dy$ aequivalere est censenda. Ac si ordine inverso primo quantitas x constans accipiatur et integrale $\int V dy$ per terminos praescriptos extendatur, id deinceps ut functio ipsius x spectari et integrale quaesitum $\int dx \int V dy$ inveniri poterit. Perinde autem est, utro modo valorem integralis formulae duplicatae $\iint V dx dy$ [evolvendi] utamur.

Cum igitur in hoc genere aliae formulae integrales nisi huiusmodi $\iint V dx dy$ occurrere nequeant, totum negotium huc redit, ut, quemadmodum huiusmodi formae variationem inveniri oporteat, ostendamus. Quoniam autem quantitates x et y variationis expertes assumimus, ex iis, quae initio sunt demonstrata [§ 75], facile colligitur fore

$$\delta \iint V dx dy = \iint \delta V dx dy,$$

ubi δV variationem ipsius V denotat; hicque integratione pariter duplici est opus, prorsus ut modo ante innuimus.

COROLLARIUM 1

160. Si ponamus integrale $\iint V dx dy = W$, cum sit $\int dx \int V dy = W$, erit per solam x differentiando $\int V dy = \left(\frac{dW}{dx}\right)$ hincque porro per y differentiando $V = \left(\frac{ddW}{dx dy}\right)$, unde patet integrale W ita comparatum esse, ut fiat $V = \left(\frac{ddW}{dx dy}\right)$.

COROLLARIUM 2

161. Cum duplex integratio sit instituenda, utraque quantitas arbitraria introducitur; altera autem integratio loco constantis functionem quamcunque ipsius x, quae sit X, altera autem functionem quamcunque ipsius y, quae sit Y, invehit, ita ut completum integrale sit

$$\iint V dx dy = W + X + Y.$$

COROLLARIUM 3

162. Hoc etiam per ipsam resolutionem confirmatur; fit enim primo

$$\int V dy = \left(\frac{dW}{dx}\right) + \left(\frac{dX}{dx}\right)$$

ob $\left(\frac{dY}{dx}\right) = 0$, tum vero fit $V = \left(\frac{ddW}{dxdy}\right)$, quia neque X neque $\frac{dX}{dx}$ ab y pendet. Quare si fuerit $\left(\frac{ddW}{dxdy}\right) = V$, erit integrale completum

$$\iint V dx dy = W + X + Y.$$

SCHOLION 1

163. Omnino autem necessarium est, ut indoles huiusmodi formularum integralium duplicatarum $\iint V dx dy$ accuratius examini subiiciatur, quod commodissime per theoriam superficierum praestari poterit. Sint ergo ut hactenus x et y binae coordinatae orthogonales in basi assumtae $AX = x$, $XY = y$ (Fig. 7)[1]), cui in Y normaliter insistat tertia ordinata $YZ = z$ ad superficiem usque porrecta. Si iam binae illae coordinatae x et y suis differentialibus crescant $XX' = dx$ et $YY' = dy$, inde in basi oritur parallelogrammum elementare $YxyY' = dxdy$, cui elementum formulae integralis convenit. Ita si de soliditate a superficie inclusa sit quaestio, eius elementum erit $= zdxdy$ ideoque tota soliditas $= \iint zdxdy$; si superficies ipsa quaeratur, posito $dz = pdx + p'dy$ erit eius elementum huic rectangulo $dxdy$ imminens $= dxdy\sqrt{(1 + pp + p'p')}$ ideoque ipsa superficies $= \iint dxdy \sqrt{(1 + pp + p'p')}$, ex quo generatim intelligitur ratio formulae integralis duplicatae $\iint V dx dy$. Quodsi iam talis formulae valor quaeratur, qui dato spatio in basi veluti $ADYX$ respondeat, primo sumta x constante investigetur integrale simplex $\int V dy$ ac tum ipsi y assignetur magnitudo XY ad curvam DY porrecta, quae ex huius curvae natura aequabitur certae functioni ipsius x. Sic igitur $dx \int V dy$ exprimet formulae propositae elementum rectangulo $XYxX' = ydx$

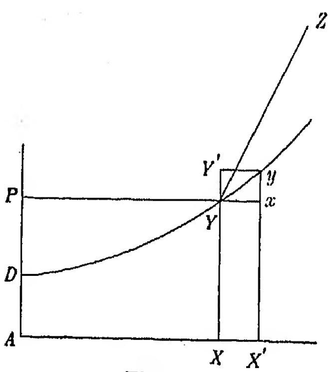

Fig. 7.

1) Fig. 7 editionis principis levi correctione concinna facta est. F. E.

conveniens, cuius integrale denuo sumtum $\int dx \int V dy$ et ex sola variabili x constans tandem dabit valorem toti spatio $ADYX$ respondentem, siquidem utraque integratio adiectione constantis rite determinetur.

SCHOLION 2

164. Ita se habere debet evolutio huiusmodi formularum integralium duplicatarum, si ad figuram in basi datam veluti $ADYX$ fuerit accommodanda; sin autem utramque integrationem indefinite expedire velimus, ut primo sumta x constante quaeramus integrale $\int V dy$, quod rectangulo elementari $XYxX' = y dx$ convenire est intelligendum, siquidem in dx ducatur, deinde vero in integratione formulae $\int dx \int V dy$ quantitatem $y = XY$ eandem manere concipiamus sola x pro variabili sumta, tum valor prodibit rectangulo indefinito $APYX = xy$ respondens, siquidem constantes per utramque integrationem ingressae debite definiantur. At si spatii istius reliqui termini praeter lineas XY et PY ut indefiniti spectentur, integrale $\iint V dx dy$ recipiet binas functiones $X + Y$ indefinitas, illam ipsius x, hanc vero ipsius y. Quodsi ergo ad calculum maximorum et minimorum haec deinceps accommodare velimus, quoniam maximi minimive proprietas, quae in spatium quodpiam datum $ADYX$ competere debet, simul quoque cuivis spatio indefinito veluti $APYX$ conveniat necesse est, duplicem illam integrationem modo hic exposito indefinito administrari conveniet.

PROBLEMA 20

165. *Si V sit formula quaecunque ex ternis variabilibus x, y, z earumque differentialibus composita, invenire variationem formulae integralis duplicatae $\iint V dx dy$, dum quantitati z, quae ut functio binarum x et y spectatur, variationes quaecunque tribuuntur.*

SOLUTIO

Ad speciem differentialium tollendam statuamus

$$p = \left(\frac{dz}{dx}\right), \quad p' = \left(\frac{dz}{dy}\right),$$

$$q = \left(\frac{dp}{dx}\right), \quad q' = \left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right), \quad q'' = \left(\frac{dp'}{dy}\right),$$

$$r = \left(\frac{dq}{dx}\right), \quad r' = \left(\frac{dq}{dy}\right) = \left(\frac{dq'}{dx}\right), \quad r'' = \left(\frac{dq'}{dy}\right) = \left(\frac{dq''}{dx}\right), \quad r''' = \left(\frac{dq''}{dy}\right)$$

etc.,

ut V fiat functio quantitatum finitarum $x, y, z, p, p', q, q', q'', r, r', r'', r'''$ etc. Tum ponatur eius differentiale

$$\begin{aligned} dV = Ldx + Mdy + Ndz + Pdp &+ Qdq + Rdr \\ + P'dp' &+ Q'dq' + R'dr' \\ &+ Q''dq'' + R''dr'' \\ &\qquad\quad + R'''dr''' \\ &\text{etc.}; \end{aligned}$$

ex quo cum simul eius variatio δV innotescat, ex problemate praecedente colligitur variatio quaesita

$$\delta\iint Vdxdy = \iint dxdy \left\{ \begin{array}{llll} N\delta z + P\delta p & + Q\delta q & + R\delta r & \\ \quad + P'\delta p' & + Q'\delta q' & + R'\delta r' & \\ & + Q''\delta q'' & + R''\delta r'' & \\ & & + R'''\delta r''' & \\ & \text{etc.} & & \end{array} \right\}$$

Quodsi iam, uti § 154 fecimus, ponamus variationem $\delta z = \omega$, quam ut functionem quamcunque binarum variabilium x et y spectare licet, indidem istam variationem concludimus fore

$$\delta\iint Vdxdy = \iint dxdy \left\{ \begin{array}{llll} N\omega + P\left(\frac{d\omega}{dx}\right) & + Q\left(\frac{dd\omega}{dx^2}\right) & + R\left(\frac{d^3\omega}{dx^3}\right) & \\ \quad + P'\left(\frac{d\omega}{dy}\right) & + Q'\left(\frac{dd\omega}{dxdy}\right) & + R'\left(\frac{d^3\omega}{dx^2dy}\right) & \\ & + Q''\left(\frac{dd\omega}{dy^2}\right) & + R''\left(\frac{d^3\omega}{dxdy^2}\right) & \\ & & + R'''\left(\frac{d^3\omega}{dy^3}\right) & \\ & \text{etc.} & & \end{array} \right\}$$

COROLLARIUM 1

166. Si ergo utriusque functionis z et $\delta z = \omega$ indoles seu ratio compositionis ex binis variabilibus x et y esset data, tum per praecepta ante

exposita variatio formulae integralis duplicatae $\iint Vdxdy$ assignari posset, quomodocunque quantitas V ex variabilibus x, y, z earumque differentialibus fuerit conflata.

COROLLARIUM 2

167. Totum scilicet negotium redibit ad evolutionem formulae integralis duplicatae inventae; quae cum pluribus constet partibus, singulas partes per duplicem integrationem, uti ante explicatum, tractari conveniet.

SCHOLION

168. Quando autem ratio functionis z non constat eaque demum ex conditione variationis elici debet, ita ut ipsa variatio $\delta z = \omega$ nullam plane determinationem patiatur, quemadmodum fit, si formula $\iint Vdxdy$ valorem maximum minimumve obtinere debeat, tum omnino necessarium est, ut singula variationis inventae $\delta\iint Vdxdy$ membra ita reducantur, ut ubique post signum integrationis duplicatum non valores differentiales variationis $\delta z = \omega$, sed haec ipsa variatio occurrat; cuiusmodi reductione iam supra in formulis binas tantum variabiles involventibus sumus usi. Talis autem reductio, cum pro formulis integralibus duplicatis minus sit consueta, accuratiorem explicationem postulat. Quem in finem observo huiusmodi reductione perveniri ad formulas simpliciter integrales, in quibus alterutra tantum quantitatum x et y pro variabili habeatur altera ut constante spectata, ad quod indicandum, ne signa praeter necessitatem multiplicentur, talis forma $\int Tdx$ denotabit integrale formulae differentialis Tdx, dum quantitas y pro constanti habetur; similique modo intelligendum est in hac forma $\int Tdy$ solam quantitatem y ut variabilem considerari, quod eo magis perspicuum est, cum hac conditione omissa hae formulae nullum plane significatum essent habiturae. Neque ergo in posterum opus est declarari, si T ambas variabiles x et y complectatur, utra earum in formulis integralibus simplicibus $\int Tdx$ vel $\int Tdy$ sive constans sive variabilis accipiatur, cum ea sola, cuius differentiale exprimitur, pro variabili sit habenda. In formulis autem duplicatis $\iint Vdxdy$ perpetuo tenendum est alteram integrationem ad solius x, alteram vero ad solius y variabilitatem adstringi perindeque esse, utra integratio prior instituatur.

PROBLEMA 21

169. *Variationem formulae integralis duplicatae $\int\int V dx dy$ in praecedente problemate inventam ita transformare, ut post signum integrale duplicatum ubique ipsa variatio $\delta z = \omega$ occurrat exturbatis eius differentialibus.*

SOLUTIO

Quo haec transformatio latius pateat, sint T et v functiones quaecunque binarum variabilium x et y et consideretur haec formula duplicata $\int\int T dx dy \left(\frac{dv}{dx}\right)$, quae separata integrationum varietate ita repraesentetur $\int dy \int T dx \left(\frac{dv}{dx}\right)$, ut in integratione $\int T dx \left(\frac{dv}{dx}\right)$ sola quantitas x ut variabilis spectetur. Tum autem erit $dx\left(\frac{dv}{dx}\right) = dv$, quia y pro constante habetur, ideoque fiet

$$\int T dv = Tv - \int v dT;$$

ubi cum in differentiali dT solius variabilis x ratio habetur, ad hoc declarandum loco dT scribi convenit $dx\left(\frac{dT}{dx}\right)$, ita ut sit

$$\int T dx \left(\frac{dv}{dx}\right) = Tv - \int v dx \left(\frac{dT}{dx}\right)$$

hincque nostra formula ita prodeat reducta

$$\int\int T dx dy \left(\frac{dv}{dx}\right) = \int Tv dy - \int\int v dx dy \left(\frac{dT}{dx}\right).$$

Simili modo permutatis variabilibus consequemur

$$\int\int T dx dy \left(\frac{dv}{dy}\right) = \int Tv dx - \int\int v dx dy \left(\frac{dT}{dy}\right).$$

Hoc iam quasi lemmate praemisso variationis in praecedente problemate inventae reductio ita se habebit

$$\int\int P dx dy \left(\frac{d\omega}{dx}\right) = \int P\omega dy - \int\int \omega dx dy \left(\frac{dP}{dx}\right),$$

$$\int\int P' dx dy \left(\frac{d\omega}{dy}\right) = \int P'\omega dx - \int\int \omega dx dy \left(\frac{dP'}{dy}\right).$$

Porro pro sequentibus membris sit primo $\left(\frac{d\omega}{dx}\right) = v$ ideoque $\left(\frac{dd\omega}{dx^2}\right) = \left(\frac{dv}{dx}\right)$, unde colligitur

$$\iint Q\,dx\,dy\left(\frac{dd\omega}{dx^2}\right)=\int Q\,dy\left(\frac{d\omega}{dx}\right)-\iint dx\,dy\left(\frac{dQ}{dx}\right)\left(\frac{d\omega}{dx}\right),$$

ac postremo membro similiter reducto fit

$$\iint Q\,dx\,dy\left(\frac{dd\omega}{dx^2}\right)=\int Q\,dy\left(\frac{d\omega}{dx}\right)-\int\omega\,dy\left(\frac{dQ}{dx}\right)+\iint\omega\,dx\,dy\left(\frac{ddQ}{dx^2}\right).$$

Per eandem substitutionem habebimus $\left(\frac{dd\omega}{dx\,dy}\right)=\left(\frac{dv}{dy}\right)$ hincque

$$\iint Q'\,dx\,dy\left(\frac{dd\omega}{dx\,dy}\right)=\int Q'\,dx\left(\frac{d\omega}{dx}\right)-\iint dx\,dy\left(\frac{d\omega}{dx}\right)\left(\frac{dQ'}{dy}\right)$$

seu

$$\iint Q'\,dx\,dy\left(\frac{dd\omega}{dx\,dy}\right)=\int Q'\,dx\left(\frac{d\omega}{dx}\right)-\int\omega\,dy\left(\frac{dQ'}{dy}\right)+\iint\omega\,dx\,dy\left(\frac{ddQ'}{dx\,dy}\right),$$

quae forma ob

$$\int Q'\,dx\left(\frac{d\omega}{dx}\right)=Q'\omega-\int\omega\,dx\left(\frac{dQ'}{dx}\right)$$

abit in hanc

$$\iint Q'\,dx\,dy\left(\frac{dd\omega}{dx\,dy}\right)=Q'\omega-\int\omega\,dx\left(\frac{dQ'}{dx}\right)+\iint\omega\,dx\,dy\left(\frac{ddQ'}{dx\,dy}\right)-\int\omega\,dy\left(\frac{dQ'}{dy}\right).$$

Tum vero pro tertia forma huius ordinis nanciscimur

$$\iint Q''\,dx\,dy\left(\frac{dd\omega}{dy^2}\right)=\int Q''\,dx\left(\frac{d\omega}{dy}\right)-\int\omega\,dx\left(\frac{dQ''}{dy}\right)+\iint\omega\,dx\,dy\left(\frac{ddQ''}{dy^2}\right).$$

Porro ob $\left(\frac{d^3\omega}{dx^3}\right)=\left(\frac{ddv}{dx^2}\right)$ manente $v=\left(\frac{d\omega}{dx}\right)$ fiet

$$\iint R\,dx\,dy\left(\frac{ddv}{dx^2}\right)=\int R\,dy\left(\frac{dv}{dx}\right)-\int v\,dy\left(\frac{dR}{dx}\right)+\iint v\,dx\,dy\left(\frac{ddR}{dx^2}\right)$$

et

$$\iint v\,dx\,dy\left(\frac{ddR}{dx^2}\right)=\int\omega\,dy\left(\frac{ddR}{dx^2}\right)-\iint\omega\,dx\,dy\left(\frac{d^3R}{dx^3}\right),$$

ita ut sit

$$\iint R\,dx\,dy\left(\frac{d^3\omega}{dx^3}\right)=\int R\,dy\left(\frac{dd\omega}{dx^2}\right)-\int dy\left(\frac{d\omega}{dx}\right)\left(\frac{dR}{dx}\right)+\int\omega\,dy\left(\frac{ddR}{dx^2}\right)$$
$$-\iint\omega\,dx\,dy\left(\frac{d^3R}{dx^3}\right).$$

Deinde ob $\left(\frac{d^3\omega}{dx^2\,dy}\right)=\left(\frac{ddv}{dx\,dy}\right)$ erit

$$\iint R'\,dx\,dy\left(\frac{ddv}{dx\,dy}\right)=R'v-\int v\,dx\left(\frac{dR'}{dx}\right)+\iint v\,dx\,dy\left(\frac{ddR'}{dx\,dy}\right)-\int v\,dy\left(\frac{dR'}{dy}\right),$$

et quia hic

$$\iint v\,dx\,dy\left(\frac{ddR'}{dx\,dy}\right)=\int\omega\,dy\left(\frac{ddR'}{dx\,dy}\right)-\iint\omega\,dx\,dy\left(\frac{d^3R'}{dx^2dy}\right),$$

concludimus fore

$$\iint R'dx\,dy\left(\frac{d^3\omega}{dx^2dy}\right)=R'\left(\frac{d\omega}{dx}\right)-\int\left(\frac{d\omega}{dx}\right)dx\left(\frac{dR'}{dx}\right)+\int\omega\,dy\left(\frac{ddR'}{dx\,dy}\right)$$
$$-\int\left(\frac{d\omega}{dx}\right)dy\left(\frac{dR'}{dy}\right)-\iint\omega\,dx\,dy\left(\frac{d^3R'}{dx^2dy}\right).$$

Tandem permutandis x et y hinc colligimus

$$\iint R''dx\,dy\left(\frac{d^3\omega}{dx\,dy^2}\right)=R''\left(\frac{d\omega}{dy}\right)-\int\left(\frac{d\omega}{dy}\right)dy\left(\frac{dR''}{dy}\right)+\int\omega\,dx\left(\frac{ddR''}{dx\,dy}\right)$$
$$-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR''}{dx}\right)-\iint\omega\,dx\,dy\left(\frac{d^3R''}{dx\,dy^2}\right).$$

et

$$\iint R'''dx\,dy\left(\frac{d^3\omega}{dy^3}\right)=\int R'''dx\left(\frac{dd\omega}{dy^2}\right)-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR'''}{dy}\right)+\int\omega\,dx\left(\frac{ddR'''}{dy^2}\right)$$
$$-\iint\omega\,dx\,dy\left(\frac{d^3R'''}{dy^3}\right).$$

Quos valores si substituamus, reperimus

$$\delta\iint V\,dx\,dy=\iint\omega\,dx\,dy\left\{\begin{aligned}&N-\left(\frac{dP}{dx}\right)+\left(\frac{ddQ}{dx^2}\right)-\left(\frac{d^3R}{dx^3}\right)\\&\quad-\left(\frac{dP'}{dy}\right)+\left(\frac{ddQ'}{dx\,dy}\right)-\left(\frac{d^3R'}{dx^2dy}\right)\\&\qquad\qquad+\left(\frac{ddQ''}{dy^2}\right)-\left(\frac{d^3R''}{dx\,dy^2}\right)\\&\qquad\qquad\qquad\qquad-\left(\frac{d^3R'''}{dy^3}\right)\end{aligned}\right\}$$

$$+\int P\omega\,dy+\int P'\omega\,dx+\int Q\,dy\left(\frac{d\omega}{dx}\right)-\int\omega\,dx\left(\frac{dQ'}{dx}\right)+Q'\omega+\int Q''dx\left(\frac{d\omega}{dy}\right)$$
$$-\int\omega\,dy\left(\frac{dQ}{dx}\right)-\int\omega\,dy\left(\frac{dQ'}{dy}\right)\qquad-\int\omega\,dx\left(\frac{dQ''}{dy}\right)$$
$$+\int R\,dy\left(\frac{dd\omega}{dx^2}\right)\quad-\int\left(\frac{d\omega}{dx}\right)dx\left(\frac{dR'}{dx}\right)+R'\left(\frac{d\omega}{dx}\right)-\int\left(\frac{d\omega}{dy}\right)dy\left(\frac{dR''}{dy}\right)+\int R'''dx\left(\frac{dd\omega}{dy^2}\right)$$
$$-\int\left(\frac{d\omega}{dx}\right)dy\left(\frac{dR}{dx}\right)-\int\left(\frac{d\omega}{dx}\right)dy\left(\frac{dR'}{dy}\right)\qquad-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR''}{dx}\right)-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR'''}{dy}\right)$$
$$+\int\omega\,dy\left(\frac{ddR}{dx^2}\right)\quad+\int\omega\,dy\left(\frac{ddR'}{dx\,dy}\right)\quad+R''\left(\frac{d\omega}{dy}\right)+\int\omega\,dx\left(\frac{ddR''}{dx\,dy}\right)\quad+\int\omega\,dx\left(\frac{ddR'''}{dy^2}\right)$$

etc.

COROLLARIUM 1

170. Huius expressionis pars prima satis est perspicua, reliquae vero partes commode ita disponi possunt, ut earum ratio comprehendatur:

$$\int \omega\, dy \left\{ \begin{array}{l} P - \left(\frac{dQ}{dx}\right) + \left(\frac{ddR}{dx^2}\right) - \text{etc.} \\ \quad - \left(\frac{dQ'}{dy}\right) + \left(\frac{ddR'}{dx\,dy}\right) \\ \qquad\qquad + \left(\frac{ddR''}{dy^2}\right) \end{array} \right\} + \int \omega\, dx \left\{ \begin{array}{l} P' - \left(\frac{dQ''}{dy}\right) + \left(\frac{ddR'''}{dy^2}\right) - \text{etc.} \\ \quad - \left(\frac{dQ'}{dx}\right) + \left(\frac{ddR''}{dx\,dy}\right) \\ \qquad\qquad + \left(\frac{ddR'}{dx^2}\right) \end{array} \right\}$$

$$+ \int \left(\frac{d\omega}{dx}\right) dy \left\{ \begin{array}{l} Q - \left(\frac{dR}{dx}\right) + \text{etc.} \\ \quad - \left(\frac{dR'}{dy}\right) \end{array} \right\} + \int \left(\frac{d\omega}{dy}\right) dx \left\{ \begin{array}{l} Q'' - \left(\frac{dR'''}{dy}\right) + \text{etc.} \\ \quad - \left(\frac{dR''}{dx}\right) \end{array} \right\}$$

$$+ \int \left(\frac{dd\omega}{dx^2}\right) dy (R - \text{etc.}) + \int \left(\frac{dd\omega}{dy^2}\right) dx (R''' - \text{etc.}) + \text{etc.}$$

$$+ \omega \left\{ \begin{array}{l} Q' - \left(\frac{dR'}{dx}\right) + \text{etc.} \\ \quad - \left(\frac{dR''}{dy}\right) \end{array} \right\} + \left(\frac{d\omega}{dx}\right)(R' - \text{etc.}) + \left(\frac{d\omega}{dy}\right)(R'' - \text{etc.}) + \text{etc.}$$

COROLLARIUM 2

171. Hic levi attentione adhibita mox patebit, quomodo istae partes ulterius continuari debeant, si forte quantitas V differentialia altiorum graduum complectatur.

COROLLARIUM 3

172. In harum formularum integralium aliis, quae differentiali dy sunt affectae, quantitas x constans sumitur, cui tribuitur valor termino integrationis conveniens; aliis vero, quae differentiali dx sunt affectae, y est constans et termino integrationis aequalis, unde patet in terminis integrationum tam x quam y recipere valorem constantem.

SCHOLION 1

173. Haec ergo variationis formula ad eum casum est accommodata, quo utriusque integrationis termini tribuunt tam ipsi x quam ipsi y valores con-

stantes. Veluti si de superficie fuerit quaestio, formula integralis $\int\int V dx dy$ ad rectangulum $APYX$ (Fig. 7, p. 460) in basi assumtum est referenda eiusque valor ita definiri debet, ut sumtis $x=0$ et $y=0$, qui sunt valores initiales, evanescat, quo facto statui oportet $x=AX$ et $y=AP$, qui sunt valores finales; atque ad eandem legem ipsa variatio inventa est expedienda. Quodsi iam ea quaeratur superficies, in qua formulae $\int\int V dx dy$ hoc modo definitae valor fiat maximus vel minimus, ante omnia necesse est, ut pars variationis prima duplicem integrationem involvens ad nihilum redigatur, quomodocunque variatio $\delta z=\omega$ accipiatur, unde haec nascetur aequatio

$$\begin{aligned} 0 = N - \left(\frac{dP}{dx}\right) + \left(\frac{ddQ}{dx^2}\right) - \left(\frac{d^3R}{dx^3}\right) + \text{etc.}, \\ - \left(\frac{dP'}{dy}\right) + \left(\frac{ddQ'}{dx\,dy}\right) - \left(\frac{d^3R'}{dx^2dy}\right) \\ + \left(\frac{ddQ''}{dy^2}\right) - \left(\frac{d^3R''}{dx\,dy^2}\right) \\ - \left(\frac{d^3R'''}{dy^3}\right) \end{aligned}$$

qua natura superficiei hac indole praeditae exprimetur. Constantes autem per duplicem integrationem ingressae ita determinari debent, ut reliquis variationis partibus satisfiat.

SCHOLION 2

174. Quo haec investigatio in se maxime abstrusa exemplo illustretur, ponamus eiusmodi superficiem investigari debere, quae inter omnes alias eandem soliditatem includentes sit minima. Hunc in finem efficiendum est, ut haec formula integralis duplicata

$$\int\int dx dy (z + a\sqrt{(1+pp+p'p')})$$

maximum minimumve evadat.[1]) Cum ergo sit

$$V = z + a\sqrt{(1+pp+p'p')},$$

erit

$$L=0, \quad M=0, \quad N=1$$

1) *Methodus inveniendi lineas curvas*, cap. V; vide notam p. 375. F. E.

atque

$$P = \frac{ap}{\sqrt{(1+pp+p'p')}} \quad \text{et} \quad P' = \frac{ap'}{\sqrt{(1+pp+p'p')}}$$

ideoque

$$dV = Ndz + Pdp + P'dp'$$

existente

$$dz = pdx + p'dy.$$

Quare superficiei quaesitae natura hac aequatione exprimetur

$$N - \left(\frac{dP}{dx}\right) - \left(\frac{dP'}{dy}\right) = 0 \quad \text{seu} \quad 1 = \left(\frac{dP}{dx}\right) + \left(\frac{dP'}{dy}\right).$$

Est vero

$$\left(\frac{dP}{dx}\right) = \frac{a}{(1+pp+p'p')^{\frac{3}{2}}}\left((1+p'p')\left(\frac{dp}{dx}\right) - pp'\left(\frac{dp'}{dx}\right)\right),$$

$$\left(\frac{dP'}{dy}\right) = \frac{a}{(1+pp+p'p')^{\frac{3}{2}}}\left((1+pp)\left(\frac{dp'}{dy}\right) - pp'\left(\frac{dp}{dy}\right)\right),$$

ubi notetur esse $\left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right)$. Ex quo ista obtinetur aequatio

$$\frac{(1+pp+p'p')^{\frac{3}{2}}}{a} = (1+p'p')\left(\frac{dp}{dx}\right) - 2pp'\left(\frac{dp}{dy}\right) + (1+pp)\left(\frac{dp'}{dy}\right);$$

quam autem quomodo tractari oporteat, haud patet, etiamsi facile perspiciatur in ea aequationem pro superficie sphaerica $zz = cc - xx - yy$, quin etiam cylindrica $zz = cc - yy$ contineri.

SVPPLEMENTVM,

CONTINENS

EVOLVTIONEM CASVVM SINGVLARIVM CIRCA INTEGRATIONEM

AEQVATIONVM DIFFERENTIALIVM.

EVOLUTIO
CASUUM PRORSUS SINGULARIUM CIRCA INTEGRATIONEM AEQUATIONUM DIFFERENTIALIUM

1. Cum adhuc plurimae atque inter se maxime discrepantes methodi sint in medium allatae aequationes differentiales integrandi, quaestio exoritur summi sane momenti, an non unica detur eaque aequabilis methodus, cuius ope omnes illas diversas aequationes differentiales, quas etiamnum resolvere licuit, integrari queant. Nullum enim est dubium, quin inventio talis methodi maxima incrementa in universam Analysin esset allatura.

Pluribus Geometris quidem separatio binarum variabilium huiusmodi methodum suppeditare est visa, cum omnes aequationum differentialium integrationes vel hac ratione sint integratae vel eo facile possint revocari. Praeterquam autem quod haec methodus substitutionibus absolvitur, quae plerumque non minorem sagacitatem postulant quam id ipsum, quod quaeritur, ac nonnunquam soli casui deberi videntur, haec methodus etiam neutiquam extenditur ad aequationes differentiales secundi altiorumque graduum; et qui tales aequationes adhuc tractaverunt, longe alia artificia in subsidium vocare sunt coacti. Quamobrem separationem variabilium nequaquam tanquam methodum uniformem ac latissime patentem spectare licet, quae omnes integrationes, quae adhuc successerunt, in se complectatur.

2. Talem autem methodum universalem iam pridem[1]) mihi equidem indicasse videor, dum ostendi proposita quacunque aequatione differentiali sive

1) Vide L. Euleri Commentationem 44 (indicis Enestroemiani): *De infinitis curvis eiusdem generis. Seu methodus inveniendi aequationes pro infinitis curvis eiusdem generis*, Comment. acad. sc. Petrop. 7 (1734/5), 1740, p. 174 paginaturae primae, p. 180 paginaturae alterius, imprimis

primi sive altioris gradus[1]) semper dari eiusmodi quantitatem, per quam si aequatio multiplicetur, evadat integrabilis, ita ut hoc modo nulla plane substitutione alibi anxie quaerenda sit opus. Ex quo non dubito hanc methodum aequationes differentiales ope multiplicationum ad integrabilitatem revocandi tanquam latissime patentem atque naturae maxime convenientem pronunciare, cum nulla integratio adhuc sit expedita, quae hoc modo non facile absolvi possit. Cum scilicet omnis aequatio differentialis primi gradus in hac forma $Pdx + Qdy = 0$ contineatur denotantibus litteris P et Q functiones quascunque binarum variabilium x et y, semper datur eiusmodi multiplicator M, itidem functio quaedam ambarum variabilium x et y, ut facta multiplicatione haec forma $MPdx + MQdy$ fiat integrabilis; cuius propterea integrale quantitati constanti arbitrariae aequatum exhibebit aequationem integralem aequationis differentialis propositae $Pdx + Qdy = 0$, quae eadem ratio quoque in aequationibus differentialibus altiorum graduum locum habet. Verum hoc argumentum hic fusius exponere non est animus, sed potius praestantiam huius methodi prae separatione variabilium etiam eiusmodi casibus, quibus id minime videatur, simulque summam eius utilitatem hic declarare constitui.

3. Quoties scilicet in aequatione differentiali variabiles x et y iam sunt separatae, totum negotium vulgo ut iam confectum spectari solet, quandoquidem huius aequationis

$$Xdx + Ydy = 0,$$

ubi X denotat functionem solius x et Y solius y, integrale in promtu est

$$\int Xdx + \int Ydy = \text{Const.}$$

Interim tamen saepenumero usu venire potest, ut hoc pacto neutiquam forma integralis simplicissima obtineatur vel ea demum per plures ambages inde derivari debeat. Veluti ex hac aequatione

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

§ 36 et 38; *Leonhardi Euleri Opera omnia*, series I, vol. 22. Cf. quoque eiusdem voluminis Commentationem 269 et voluminis 23 Commentationes 429, 650, 720. Vide etiam *Institutionum calculi integralis* vol. I, § 443—530, vol. II, § 865—927, *Leonhardi Euleri Opera omnia*, series I, vol. 11 et 12. L. S.

1) Supplendum est: „*cuius integratio adhuc successit*". L. S.

primo elicitur integrale logarithmicum

$$lx + ly = la,$$

unde quidem statim se prodit algebraicum $xy = a$. Verum ex hac forma

$$\frac{dx}{aa + xx} + \frac{dy}{aa + yy} = 0$$

integratio solita praebet

$$\text{Ang. tang. } \frac{x}{a} + \text{Ang. tang. } \frac{y}{a} = \text{Const.}^{1)},$$

unde non tam facile forma integralis algebraica $\frac{x+y}{aa - xy} = C$ deducitur. Ac proposita hac forma

$$\frac{dx}{\sqrt{(\alpha + \beta x + \gamma xx)}} + \frac{dy}{\sqrt{(\alpha + \beta y + \gamma yy)}} = 0$$

in genere ne patet quidem, utrum utraque pars integralis arcu circulari an logarithmo exprimatur. Interim tamen eius integrale ita algebraice exhiberi potest [2])

$$CC(x - y)^2 + 2\gamma Cxy + \beta C(x + y) + 2\alpha C + \tfrac{1}{4}\beta\beta - \alpha\gamma = 0,$$

quae certe forma simplicissima nonnisi per plures ambages ex integrali transcendente derivatur.

4. His quidem casibus perspicitur, quomodo reductionem ad formam algebraicam institui oporteat; sed ante aliquot annos eiusmodi integrationes protuli [3]), in quibus ne hoc quidem ullo modo praestari potest. Veluti si

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1) Editio princeps: Ang. tang. x + Ang. tang. y = Const. Correxit L. S.

2) Vide *Institutionum calculi integralis* vol. I, § 580, *Leonhardi Euleri Opera omnia*, series I, vol. 11. L. S.

3) Vide L. Euleri Commentationes 211, 251, 252, 263, 261, 264, 345 (indicis Eneströmiani), *Leonhardi Euleri Opera omnia*, series I, vol. 20. Cf. etiam voluminis 21 Commentationes 506, 581, 582, 605, 676, 714, 818; porro (ad Commentationem 506) I. L. de Lagrange, *Sur l'intégration de quelques équations différentielles dont les indéterminés sont séparées, mais dont chaque membre en particulier n'est point intégrable*, Miscellanea Taurin. 4, 1766/9, II, p. 98; *Oeuvres de Lagrange*, t. II, p. 5. In Commentatione 251 Eulerus ipse ait se has aequationes integrales algebraicas tum primum contemplatum esse, cum anno 1751 mandatu Academiae Berolinensis de opere Comitis I. C. Fagnano, quod inscribitur *Produzioni matematiche* (Pesaro, 1750), retulisset. L. S.

proposita sit haec aequatio[1])

$$\frac{dx}{\sqrt{(1+x^4)}}+\frac{dy}{\sqrt{(1+y^4)}}=0,$$

integrationem neque per logarithmos neque arcus circulares expedire licet, ut inde deinceps simili ratione aequatio algebraica colligi posset; interim tamen ostendi huius integrale idque adeo completum hoc modo algebraice exprimi

$$0=2C+(CC-1)(xx+yy)-2(1+CC)xy+2Cxxyy,$$

ubi C denotat constantem per integrationem ingressam. Quin etiam huius aequationis multo latius patentis

$$\frac{dx}{\sqrt{(\alpha+2\beta x+\gamma xx+2\delta x^3+\varepsilon x^4)}}+\frac{dy}{\sqrt{(\alpha+2\beta y+\gamma yy+2\delta y^3+\varepsilon y^4)}}=0$$

integrale completum est[2])

$$\begin{aligned}0&=2\alpha C+\beta\beta-\alpha\gamma+2(\beta C-\alpha\delta)(x+y)+(CC-\alpha\varepsilon)(xx+yy)\\&+2(\gamma C-CC-\alpha\varepsilon-\beta\delta)xy+2(\delta C-\beta\varepsilon)xy(x+y)+(2\varepsilon C+\delta\delta-\gamma\varepsilon)xxyy\end{aligned}$$

denotante C item constantem quantitatem arbitrariam per integrationem inventam. His igitur casibus perspicuum est separationem variabilium, qua aequationes differentiales sunt praeditae, nihil plane iuvare ad integralia earum forma algebraica contenta eruenda, ex quo merito eiusmodi methodus desideratur, cuius beneficio haec integralia statim ex aequationibus differentialibus investigari potuissent, in quo negotio certe omnes ingenii vires tentasse non pigebit.

5. Observavi igitur hunc scopum ope multiplicatorum idoneorum obtineri posse, quibus aequationes differentiales multiplicatae ita integrabiles evadant, ut integralia statim algebraice expressa prodeant. Quod quo clarius perspiciatur, ab aequatione primum proposita $\frac{dx}{x}+\frac{dy}{y}=0$ exordiar, quae per xy multiplicata statim praebet $ydx+xdy=0$, cuius integrale est $xy=C$.

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1) Vide L. EULERI Commentationem 347 (indicis ENESTROEMIANI), *Leonhardi Euleri Opera omnia*, series I, vol. 20, p. 338, porro *Institutionum calculi integralis* vol. I, § 632, *Leonhardi Euleri Opera omnia*, series I, vol. 11. L. S.

2) Vide § 22 huius Supplementi. L. S.

Hoc ergo modo sublata separatione aequatio in aliam transformatur, quae integrationem admittit, ex quo intelligitur methodum ope multiplicatorum integrandi id praestare, quod a separatione variabilium immediate expectari nequeat. Idem evenit in aequatione $\frac{m\,dx}{x}+\frac{n\,dy}{y}=0$, quae per $x^m y^n$ multiplicata integrale praebet $x^m y^n = C$, dum ex ipsa aequatione proposita statim ad logarithmos fuisset perventum. Simili modo si haec aequatio separata

$$\frac{dx}{1+xx}+\frac{dy}{1+yy}=0$$

multiplicetur in $\frac{(1+xx)(1+yy)}{(x+y)^2}$, aequatio resultans

$$\frac{dx(1+yy)+dy(1+xx)}{(x+y)^2}=0$$

integrationem iam sponte admittit praebetque integrata

$$\frac{-1+xy}{x+y}=\text{Const.}\quad\text{seu}\quad\frac{x+y}{1-xy}=a.$$

Hanc vero aequationem

$$\frac{2\,dx}{1+xx}+\frac{dy}{1+yy}=0$$

multiplicari convenit in $\frac{(xx+1)^2(1+yy)}{(2xy+xx-1)^2}$, ut prodeat

$$\frac{2\,dx(1+xx)(1+yy)+dy(xx+1)^2}{(2xy+xx-1)^2}=0,$$

cuius integrale reperitur

$$\frac{xxy-2x-y}{2xy+xx-1}=\text{Const.}\quad\text{seu}\quad\frac{2x+y-xxy}{2xy+xx-1}=a.$$

6. Contra haec exempla, quibus integralia algebraica sine subsidio separationis sunt eruta, obiicietur multiplicatores negotium hoc conficientes ex ipsis integralibus illis transcendentibus, ad quae separatio variabilium immediate perducit, esse conclusos iisque adeo praestantiam methodi per multiplicatores procedentis neutiquam probari. Cui quidem obiectioni primum respondeo priora exempla statim ab inventis integrationis principiis simili modo fuisse expedita, antequam integratio per logarithmos erat explorata, quae

ergo nullum subsidium eo attulisse est censenda. Tum vero, quamvis concedam in posterioribus exemplis integrationem per arcus circulares multiplicatores illos idoneos commode suppeditasse, id tamen in ipsa evolutione minus cernitur eademque integratio sine dubio inveniri potuisset, antequam constaret formulae $\frac{dx}{1+xx}$ integrale esse arcum circuli tangenti x respondentem. Verum aequatio supra [§ 4] allata

$$\frac{dx}{\sqrt{(1+x^4)}}+\frac{dy}{\sqrt{(1+y^4)}}=0,$$

cuius integrale completum algebraice exhibere licet, nulli amplius dubio locum relinquit; cum enim neutrius partis integrale ne concessis quidem logarithmis vel arcubus circularibus exhiberi possit eiusque forma ad genus quantitatum transcendentium etiamnum incognitum sit referenda, haec certe nullum auxilium ad integrale algebraicum inveniendum attulisse censeri potest. Atque hoc multo magis de aequatione illa latius patente in § 4 proposita est tenendum, quippe cuius integratio omnino singularis ex principiis longe diversissimis a me est eruta.

7. Methodus autem, qua tum sum usus, tantopere est abscondita, ut vix ulla via ad eadem integralia perducens patere videatur, et cum separatio variabilium nihil plane eo contulisset, vix etiam quicquam ab altera methodo ad multiplicatores adstricta sperari posse videbatur, propterea quod tum ipse adhuc in ea opinione versabar per multiplicatores nihil praestari posse, nisi quatenus separatio variabilium eodem manuducat, quandoquidem quaestio differentialia tantum primi gradus implicaret. Deinceps autem re diligentius considerata perspexi, quoties aequationis cuiusque differentialis integrale completum exhibere licet, ex eo vicissim semper eiusmodi multiplicatorem elici posse, per quem si aequatio differentialis multiplicetur, non solum fiat integrabilis, sed etiam integrata id ipsum integrale, quod iam erat cognitum, reproducere debeat; ad hoc autem omnino necesse est, ut integrale completum sit exploratum, dum ex integralibus particularibus nihil plane pro hoc scopo concludere licet.

Si enim proposita sit aequatio differentialis

$$Pdx+Qdy=0,$$

cuius integrale completum undecunque sit cognitum, constabit id aequatione,

quae praeter binas variabiles x et y et quantitates constantes in ipsa aequatione differentiali contentas insuper quantitatem constantem novam prorsus ab arbitrio nostro pendentem complectetur. Quae si littera C indicetur, eruatur eius valor ex aequatione integrali ac reperiatur $C = V$ eritque V certa quaedam functio ipsarum x et y; tum autem hac aequatione differentiata $0 = dV$ differentiale dV necessario ita formulam differentialem $Pdx + Qdy$ continere debet, ut sit

$$dV = M(Pdx + Qdy),$$

ex qua forma multiplicator M ad hoc integrale $C = V$ perducens sponte se offert.

8. Quo haec operatio aliquot exemplis illustretur, sumatur primo haec aequatio

$$\frac{m\,dx}{x} + \frac{n\,dy}{y} = 0;$$

cuius integrale completum cum sit $x^m y^n = C$, instituta differentiatione prodit

$$0 = mx^{m-1}y^n dx + nx^m y^{n-1} dy \quad \text{seu} \quad 0 = x^m y^n\left(\frac{m\,dx}{x} + \frac{n\,dy}{y}\right),$$

unde patet multiplicatorem ad hoc integrale ducentem esse $x^m y^n$.

Deinde cum huius aequationis

$$\frac{dx}{1+xx} + \frac{dy}{1+yy} = 0$$

integrale completum sit $1 - xy = C(x+y)$, valor constantis arbitrariae hinc fit $C = \frac{1-xy}{x+y}$, cuius differentiatio praebet

$$0 = \frac{-dx(1+yy) - dy(1+xx)}{(x+y)^2} \quad \text{seu} \quad 0 = \frac{(1+xx)(1+yy)}{(x+y)^2}\left(\frac{dx}{1+xx} + \frac{dy}{1+yy}\right),$$

unde multiplicator quaesitus est $= \frac{(1+xx)(1+yy)}{(x+y)^2}$.

Proposita porro sit haec aequatio

$$\frac{dx}{\sqrt{(\alpha + 2\beta x + \gamma xx)}} + \frac{dy}{\sqrt{(\alpha + 2\beta y + \gamma yy)}} = 0,$$

cuius integrale completum

$$CC(x-y)^2 - 2C(\alpha + \beta x + \beta y + \gamma xy) + \beta\beta - \alpha\gamma = 0$$

dat primo [1])

$$C = \frac{\alpha + \beta(x+y) + \gamma xy + \sqrt{(\alpha\alpha + 2\alpha\beta(x+y) + \alpha\gamma(xx+yy) + 4\beta\beta xy + 2\beta\gamma xy(x+y) + \gamma\gamma xxyy)}}{(x-y)^2}$$

seu [1])

$$C = \frac{\alpha + \beta(x+y) + \gamma xy + \sqrt{(\alpha + 2\beta x + \gamma xx)(\alpha + 2\beta y + \gamma yy)}}{(x-y)^2}$$

vel concinnius

$$\frac{\beta\beta - \alpha\gamma}{C} = \alpha + \beta(x+y) + \gamma xy + \sqrt{(\alpha + 2\beta x + \gamma xx)(\alpha + 2\beta y + \gamma yy)},$$

unde differentiando fit

$$0 = dx(\beta + \gamma y) + dy(\beta + \gamma x)$$
$$+ \frac{dx(\beta + \gamma x)\sqrt{(\alpha + 2\beta y + \gamma yy)}}{\sqrt{(\alpha + 2\beta x + \gamma xx)}} + \frac{dy(\beta + \gamma y)\sqrt{(\alpha + 2\beta x + \gamma xx)}}{\sqrt{(\alpha + 2\beta y + \gamma yy)}}$$

hincque colligitur multiplicator quaesitus

$$M = (\beta + \gamma x)\sqrt{(\alpha + 2\beta y + \gamma yy)} + (\beta + \gamma y)\sqrt{(\alpha + 2\beta x + \gamma xx)}.$$

9. Simili modo pro aequatione magis complexa

$$\frac{dx}{\sqrt{(\alpha + 2\beta x + \gamma xx + 2\delta x^3 + \varepsilon x^4)}} + \frac{dy}{\sqrt{(\alpha + 2\beta y + \gamma yy + 2\delta y^3 + \varepsilon y^4)}} = 0$$

ex eius integrali completo supra [§ 4] exhibito multiplicator idoneus M investigari poterit, ex quo, si statim fuisset cognitus, idem hoc integrale immediate elici potuisset. Verum hic opus multo maius molior, quod autem primo conatu neutiquam ad finem perducere licebit; ex quo satis mihi equidem praestitisse videbor, si saltem prima quasi lineamenta novae atque maxime desiderandae methodi adumbravero, cuius ope proposita huiusmodi aequatione differentiali multiplicator idoneus eam reddens integrabilem inveniri queat.

1) In his duabus aequationibus revera signum + ante radicem in — convertendum est, ut aequationes sequentes inde obtineantur. L. S.

Ac primo quidem in hoc negotio plurimum observasse iuvabit, si unicus huiusmodi multiplicator innotuerit, ex eo facile infinitos alios idem officium praestantes erui posse. Quodsi enim multiplicator M aequationem differentialem $Pdx + Qdy = 0$ integrabilem reddat, ita ut sit

$$\int M(Pdx + Qdy) = V$$

ideoque aequatio integralis $V = C$, quoniam formula

$$dV = M(Pdx + Qdy)$$

per functionem quamcunque quantitatis V multiplicata perinde manet integrabilis, perspicuum est hanc formam $Mf : V$, quaecunque functio ipsius V pro $f : V$ accipiatur, semper multiplicatorem idoneum praebere, cum sit

$$(Pdx + Qdy)Mf : V = dVf : V$$

ideoque integrabile. Inter infinitos igitur hos multiplicatores idoneos quovis casu eum eligi conveniet, qui negotium facillime conficiat et integrale, si fuerit algebraicum, forma simplicissima exhibeat. Etiamsi enim integrale revera sit algebraicum, omnino fieri potest, ut id ne suspicari quidem liceat, nisi multiplicator idoneus in usum vocetur, quemadmodum superiora exempla abunde declarant.

10. Sit ergo aequatio differentialis proposita huius formae

$$\frac{dx}{X} + \frac{dy}{Y} = 0,$$

in qua X sit functio solius x et Y solius y, atque investigari oporteat eiusmodi multiplicatorem M, quo illa aequatio algebraice integrabilis reddatur, siquidem fieri potest; quod cum raro eveniat, vicissim assumta multiplicatoris forma M indagasse iuvabit functiones X et Y.

Sit primo multiplicator

$$M = \frac{XY}{(\alpha + \beta x + \gamma y)^2},$$

ut integrabilis esse debeat haec forma

$$\frac{Ydx + Xdy}{(\alpha + \beta x + \gamma y)^2}.$$

Hinc sumta y constante colligitur integrale

$$\frac{-Y}{\beta(\alpha+\beta x+\gamma y)}+\Gamma:y,$$

sumta autem x constante prodit

$$\frac{-X}{\gamma(\alpha+\beta x+\gamma y)}+\varDelta:x,$$

quas ambas formas inter se aequales esse oportet; unde fit

$$-\gamma Y+\beta\gamma(\alpha+\beta x+\gamma y)\Gamma:y=-\beta X+\beta\gamma(\alpha+\beta x+\gamma y)\varDelta:x$$

seu

$$\beta X-\gamma Y=\beta\gamma(\alpha+\beta x+\gamma y)(\varDelta:x-\Gamma:y)$$

sicque patet functiones $\varDelta:x$ et $\Gamma:y$ ita comparatas esse debere, ut evoluto posteriori membro termini, qui simul x et y continerent, se mutuo tollant. Ex quo intelligitur fore

$$\varDelta:x=m\beta x+\text{Const.} \quad \text{et} \quad \Gamma:y=m\gamma y+\text{Const.}$$

Statuamus ergo

$$\varDelta:x-\Gamma:y=m\beta x-m\gamma y+n$$

fietque

$$\beta X-\gamma Y=\beta\gamma\left\{\begin{matrix} m\beta\beta xx-m\gamma\gamma yy+n\beta x+n\gamma y+n\alpha \\ +m\alpha\beta x-m\alpha\gamma y+f \\ -f \end{matrix}\right\},$$

unde colligimus

$$X=\gamma(m\beta\beta xx+\beta(m\alpha+n)x+f+\tfrac{1}{2}n\alpha),$$
$$Y=\beta(m\gamma\gamma yy+\gamma(m\alpha-n)y+f-\tfrac{1}{2}n\alpha),$$

et integralis aequatio algebraica erit

$$m\gamma y-\frac{m\gamma\gamma yy+\gamma(m\alpha-n)y+f-\frac{1}{2}n\alpha}{\alpha+\beta x+\gamma y}=\text{Const.}$$

seu

$$m\beta\gamma xy+n\gamma y-f+\frac{1}{2}n\alpha=C(\alpha+\beta x+\gamma y)$$

vel loco C scribendo $C+\frac{1}{2}n$ erit concinnius

$$m\beta\gamma xy-\frac{1}{2}n\beta x+\frac{1}{2}n\gamma y-f=C(\alpha+\beta x+\gamma y).$$

11. Videamus iam, sub quibus conditionibus haec forma aequationis generalis ista ratione integrabilis evadat

$$\frac{hdx}{Axx+Bx+C}+\frac{kdy}{Dyy+Ey+F}=0.$$

Comparatione ergo cum valoribus inventis instituta colligitur

$$\begin{aligned} A &= hm\beta\beta\gamma, & D &= km\beta\gamma\gamma, \\ B &= h\beta\gamma(m\alpha+n), & E &= k\beta\gamma(m\alpha-n), \\ C &= h\gamma(f+\tfrac{1}{2}n\alpha), & F &= k\beta(f-\tfrac{1}{2}n\alpha). \end{aligned}$$

Quoniam hic totum negotium ad rationes litterarum reducitur, sumtis pro primis aequalitatibus

$$\beta = Ak \quad \text{et} \quad \gamma = Dh$$

concluduntur reliquae

$$m=\frac{1}{ADhhkk}, \quad \alpha=\frac{Bk+Eh}{2}, \quad n=\frac{Bk-Eh}{2ADhhkk} \quad \text{et} \quad f=\frac{ACkk+DFhh}{2ADhhkk};$$

praeterea vero haec conditio requiritur, ut sit

$$\frac{4AC-BB}{hh}=\frac{4DF-EE}{kk};$$

quae si habuerit locum, multiplicator idoneus erit

$$M=\frac{(Axx+Bx+C)(Dyy+Ey+F)}{hk(\frac{1}{2}(Bk+Eh)+Akx+Dhy)^2}$$

et aequatio integralis inde resultans erit per hk multiplicando

$$xy-\frac{(Bk-Eh)x}{4Dh}+\frac{(Bk-Eh)y}{4Ak}-\frac{ACkk+DFhh}{2ADhk}=G\left(\tfrac{1}{2}(Bk+Eh)+Akx+Dhy\right),$$

quae immutata constante arbitraria G ad hanc formam revocatur

$$\left(x+\frac{B}{2A}-GDh\right)\left(y+\frac{E}{2D}-GAk\right)$$
$$=GGADhk+\frac{(4AC-BB)kk+(4DF-EE)hh}{8ADhk}.$$

seu

$$\left(\frac{2Ax+B}{h}+G\right)\left(\frac{2Dy+E}{k}+G\right)=GG+\frac{4AC-BB}{2hh}+\frac{4DF-EE}{2kk}.$$

12. En ergo theorema minime spernendum, etiamsi eius veritas ex aliis principiis satis manifesta esse queat.

Si haec aequatio differentialis

$$\frac{h\,dx}{Axx+Bx+C}+\frac{k\,dy}{Dyy+Ey+F}=0$$

ita fuerit comparata, ut sit

$$\frac{4AC-BB}{hh}=\frac{4DF-EE}{kk},$$

tum eius integrale completum erit algebraicum atque hac aequatione expressum

$$\frac{2Ax+B}{h}\cdot\frac{2Dy+E}{k}+G\left(\frac{2Ax+B}{h}+\frac{2Dy+E}{k}\right)=\frac{4AC-BB}{2hh}+\frac{4DF-EE}{2kk},$$

ubi G denotat constantem arbitrariam per integrationem invectam. Hoc vero integrale invenitur, si aequatio proposita ducatur in hunc multiplicatorem

$$\frac{(Axx+Bx+C)(Dyy+Ey+F)}{\left(\frac{2Ax+B}{h}+\frac{2Dy+E}{k}\right)^2}.$$

13. Quemadmodum multiplicatori M tribuimus formam

$$\frac{XY}{(\alpha+\beta x+\gamma y)^2},$$

ita etiam formis magis complicatis uti licebit, quod quidem in genere praestari nequit. Evolvamus autem multiplicatorem

$$M=\frac{YX}{(\alpha+\beta x+\gamma y+\delta xy)^2},$$

ut haec aequatio integrabilis sit efficienda

$$\frac{Ydx+Xdy}{(\alpha+\beta x+\gamma y+\delta xy)^2}=0,$$

cuius integratio ad hanc perducit aequationem

$$\frac{-Y}{(\beta+\delta y)(\alpha+\beta x+\gamma y+\delta xy)}+\Gamma:y=\frac{-X}{(\gamma+\delta x)(\alpha+\beta x+\gamma y+\delta xy)}+\Delta:x,$$

quae transformatur in hanc

$$\frac{X}{\gamma+\delta x}-\frac{Y}{\beta+\delta y}=(\alpha+\beta x+\gamma y+\delta xy)(\varDelta:x-\varGamma:y),$$

ubi evidens est statui debere

$$\varDelta:x=\frac{\zeta x+\eta}{\gamma+\delta x}\quad\text{et}\quad\varGamma:y=\frac{\zeta y+\theta}{\beta+\delta y},$$

ut nulli termini occurrant, qui utramque variabilem simul complectantur. Hinc ergo fiet

$$\frac{X}{\gamma+\delta x}-\frac{Y}{\beta+\delta y}=\begin{matrix}\eta y\\+f\end{matrix}+\frac{(\alpha+\beta x)(\zeta x+\eta)}{\gamma+\delta x}\begin{matrix}-\theta x\\-f\end{matrix}-\frac{(\alpha+\gamma y)(\zeta y+\theta)}{\beta+\delta y},$$

unde concludimus

$$X=(\alpha+\beta x)(\zeta x+\eta)-(\gamma+\delta x)(\theta x+f),$$
$$Y=(\alpha+\gamma y)(\zeta y+\theta)-(\beta+\delta y)(\eta y+f)$$

sive evolvendo

$$X=(\beta\zeta-\delta\theta)xx+(\alpha\zeta+\beta\eta-\gamma\theta-\delta f)x+\alpha\eta-\gamma f,$$
$$Y=(\gamma\zeta-\delta\eta)yy+(\alpha\zeta+\gamma\theta-\beta\eta-\delta f)y+\alpha\theta-\beta f,$$

et aequatio integralis erit

$$\frac{\zeta x+\eta}{\gamma+\delta x}-\frac{X}{(\gamma+\delta x)(\alpha+\beta x+\gamma y+\delta xy)}=\text{Const.},$$

quae loco X substituto valore invento abit in hanc formam

$$\frac{\zeta xy+\eta y+\theta x+f}{\alpha+\beta x+\gamma y+\delta xy}=\text{Const.}$$

14. Transferamus haec iterum ad formam

$$\frac{h\,dx}{Axx+Bx+C}+\frac{k\,dy}{Dyy+Ey+F}=0$$

ac fieri oportet

$$A=h(\beta\zeta-\delta\theta),\qquad D=k(\gamma\zeta-\delta\eta),$$
$$B=h(\alpha\zeta+\beta\eta-\gamma\theta-\delta f),\qquad E=k(\alpha\zeta+\gamma\theta-\beta\eta-\delta f),$$
$$C=h(\alpha\eta-\gamma f),\qquad F=k(\alpha\theta-\beta f).$$

Primae aequationes praebent

$$\theta = \frac{\beta\zeta}{\delta} - \frac{A}{\delta h}, \quad \eta = \frac{\gamma\zeta}{\delta} - \frac{D}{\delta k},$$

secundae vero

$$f = \frac{\alpha\zeta}{\delta} - \frac{Bk+Eh}{2\delta hk} \quad \text{et} \quad \delta = \frac{2A\gamma k - 2D\beta h}{Bk - Eh},$$

unde ex tertiis colligitur

$$\frac{2Ck(A\gamma k - D\beta h)}{Bk-Eh} = \frac{\gamma}{2}(Bk+Eh) - D\alpha h,$$

$$\frac{2Fh(A\gamma k - D\beta h)}{Bk-Eh} = \frac{\beta}{2}(Bk+Eh) - A\alpha k.$$

Hinc α elidendo fit

$$\frac{2(ACkk - DFhh)(Ak\gamma - Dh\beta)}{Bk-Eh} = \frac{1}{2}(Ak\gamma - Dh\beta)(Bk+Eh),$$

unde, cum esse nequeat

$$Ak\gamma - Dh\beta = 0,$$

quia alioquin fieret $\delta = 0$ et quantitates θ, η, f infinitae, tum vero, quod praecipue est notandum, aequatio integralis prodiret Const. = Const., quo ergo casu nihil indicaretur, necesse est, ut sit

$$4(ACkk - DFhh) = BBkk - EEhh$$

seu

$$\frac{4AC - BB}{hh} = \frac{4DF - EE}{kk}$$

ut ante.

Quod autem hic maxime animadverti meretur, est, quod, etsi tres litterae β, γ et ζ manent indefinitae, aequatio tamen integralis a praecedente nonnisi quantitate constante discrepat; prodit enim

$$\frac{2\zeta hk}{Bk-Eh} - \frac{k(2Ax+B)+h(2Dy+E)}{2(Ak\gamma - Dh\beta)xy + (Bk-Eh)(\beta x+\gamma y) + 2(Ck\beta - Fh\gamma)} = \text{Const.}$$

seu

$$\frac{\gamma ky(2Ax+B) + \beta k(Bx+2C) - \beta hx(2Dy+E) - \gamma h(Ey+2F)}{k(2Ax+B)+h(2Dy+E)} = \text{Const.},$$

quae forma, quomodocunque accipiantur litterae β et γ, semper veram aequationem integralem exhibet. Quod cum minus sit perspicuum, ostendisse suf-

ficiet ambas partes β et γ involventes seorsim sumtas eandem relationem inter x et y definire. Constitutis enim his duabus aequationibus

$$\frac{2Akxy+Bky-Ehy-2Fh}{2Akx+2Dhy+Bk+Eh}=\text{Const.},\qquad \frac{-2Dhxy-Ehx+Bkx+2Ck}{2Akx+2Dhy+Bk+Eh}=\text{Const.}$$

multiplicetur prior per Dh, posterior per Ak fietque summa

$$\frac{Ak(Bk-Eh)x+Dh(Bk-Eh)y+2ACkk-2DFhh}{2Akx+2Dhy+Bk+Eh},$$

cuius valor utique est constans $=\frac{Bk-Eh}{2}$, propterea quod

$$\frac{2ACkk-2DFhh}{Bk+Eh}=\frac{Bk-Eh}{2},$$

unde patet propositum.

15. Progredior nunc ad formam aequationum magis arduam, quae sit

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

sitque multiplicator eam reddens integrabilem

$$M=P\sqrt{X}+Q\sqrt{Y},$$

ita ut aequatio integrationem admittens sit

$$Pdx+Qdy+\frac{Qdx\sqrt{Y}}{\sqrt{X}}+\frac{Pdy\sqrt{X}}{\sqrt{Y}}=0,$$

cuius utrumque membrum seorsim integrabile sit oportet.

Pro priore ergo erit $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, posterioris vero integrale statuatur $2V\sqrt{XY}$, unde colligitur

$$Q=2X\left(\frac{dV}{dx}\right)+V\frac{dX}{dx}\quad\text{et}\quad P=2Y\left(\frac{dV}{dy}\right)+V\frac{dY}{dy}$$

et ob priorem conditionem

$$2Y\left(\frac{ddV}{dy^2}\right)+\frac{3dY}{dy}\left(\frac{dV}{dy}\right)+V\frac{ddY}{dy^2}=2X\left(\frac{ddV}{dx^2}\right)+\frac{3dX}{dx}\left(\frac{dV}{dx}\right)+V\frac{ddX}{dx^2},$$

ex qua aequatione, si loco V sumserimus certam functionem ipsarum x et y, dispiciendum est, quomodo idonei valores pro functionibus X et Y obtineantur.

16. Demus primo ipsi V valorem constantem, puta $V=1$, ac pervenimus ad hanc conditionem

$$\frac{ddY}{dy^2}=\frac{ddX}{dx^2},$$

quae aequalitas subsistere nequit, nisi utrumque membrum seorsim aequetur quantitati constanti, quae sit $=2a$, unde colligemus

$$X=axx+bx+c \quad \text{et} \quad Y=ayy+dy+e$$

hincque porro

$$P=\frac{dY}{dy}=2ay+d \quad \text{et} \quad Q=\frac{dX}{dx}=2ax+b,$$

unde aequatio integralis completa colligitur

$$2axy+dx+by+2\sqrt{XY}=\text{Const.}$$

Quocirca ista aequatio differentialis

$$\frac{dx}{\sqrt{(axx+bx+c)}}+\frac{dy}{\sqrt{(ayy+dy+e)}}=0$$

integrabilis redditur ope multiplicatoris

$$M=(2ay+d)\sqrt{(axx+bx+c)}+(2ax+b)\sqrt{(ayy+dy+e)}$$

ac tum integrale completum reperietur

$$2axy+dx+by+2\sqrt{(axx+bx+c)(ayy+dy+e)}=C$$

seu sublata irrationalitate

$$CC-2C(2axy+dx+by)=(4ae-dd)xx+(4ac-bb)yy+2bdxy$$
$$+4bex+4cdy+4ce.$$

Haec autem aequatio differentialis multo latius patet illa, quam initio § 3 attuleram.

17. Tribuamus nunc ipsi V hunc valorem

$$V=\frac{1}{(\alpha+\beta x+\gamma y)^2};$$

si enim loco exponentis 2 indefinitum sumsissem, mox patuisset hanc potestatem accipi debuisse. Erit ergo

$$\left(\frac{dV}{dx}\right)=\frac{-2\beta}{(\alpha+\beta x+\gamma y)^3},\quad \left(\frac{dV}{dy}\right)=\frac{-2\gamma}{(\alpha+\beta x+\gamma y)^3},$$

$$\left(\frac{ddV}{dx^2}\right)=\frac{6\beta\beta}{(\alpha+\beta x+\gamma y)^4}\quad\text{et}\quad \left(\frac{ddV}{dy^2}\right)=\frac{6\gamma\gamma}{(\alpha+\beta x+\gamma y)^4}.$$

His autem valoribus substitutis sequentes oriuntur binae formae

$$12\beta\beta X-\frac{6\beta dX}{dx}(\alpha+\beta x+\gamma y)+\frac{ddX}{dx^2}(\alpha+\beta x+\gamma y)^2$$

$$=12\gamma\gamma Y-\frac{6\gamma dY}{dy}(\alpha+\beta x+\gamma y)+\frac{ddY}{dy^2}(\alpha+\beta x+\gamma y)^2;$$

quia igitur in priore y, in altera x non ultra duas dimensiones assurgit, evidens est in formulis

$$\frac{ddX}{dx^2}\quad\text{et}\quad\frac{ddY}{dy^2}$$

variabiles x et y totidem dimensiones habere debere, quia alioquin termini ex x et y mixti utrinque aequales fieri non possent. Cum ergo ipsae functiones X et Y ad quartum gradum sint ascensurae, ponamus

$$X=Ax^4+2Bx^3+Cx^2+2Dx+E\quad\text{et}\quad Y=\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}y^2+2\mathfrak{D}y+\mathfrak{E}.$$

Facta iam substitutione pro priori parte prodit

$$\begin{array}{lllll}
12\beta\beta Ax^4 & +24\beta\beta Bx^3 & +12\beta\beta Cxx & +24\beta\beta Dx & +12\beta\beta E\\
-24\beta\beta A & -36\beta\beta B & -12\beta\beta C & -12\beta\beta D & -12\alpha\beta D\\
+12\beta\beta A & -24\alpha\beta A & -36\alpha\beta B & -12\alpha\beta C & +\ \ 2\alpha\alpha C\\
 & +12\beta\beta B & +\ \ 2\beta\beta C & +\ \ 4\alpha\beta C & \\
 & +24\alpha\beta A & +24\alpha\beta B & +12\alpha\alpha B & \\
 & & +12\alpha\alpha A & & \\
-24\beta\gamma Ax^3y & -36\beta\gamma Bx^2y & -12\beta\gamma Cxy & -12\beta\gamma Dy & \\
+24\beta\gamma A & +24\beta\gamma B & +\ \ 4\beta\gamma C & +\ \ 4\alpha\gamma C & \\
 & +24\alpha\gamma A & +24\alpha\gamma B & & \\
\multicolumn{5}{l}{+12\gamma\gamma Axxyy+12\gamma\gamma Bxyy+\ \ 2\gamma\gamma Cyy,}
\end{array}$$

qui termini in ordinem disponantur

$$\begin{aligned}&12\gamma\gamma Axxyy + 12\gamma\gamma Bxyy + 12\gamma(2\alpha A - \beta B)xxy\\ &+ 2\gamma\gamma Cyy + 8\gamma(3\alpha B - \beta C)xy + 2(6\alpha\alpha A - 6\alpha\beta B + \beta\beta C)xx\\ &+ 4\gamma(\alpha C - 3\beta D)y + 4(3\alpha\alpha B - 2\alpha\beta C + 3\beta\beta D)x + 2(\alpha\alpha C - 6\alpha\beta D + 6\beta\beta E).\end{aligned}$$

Simili vero modo altera pars erit

$$\begin{aligned}&12\beta\beta\mathfrak{A}xxyy + 12\beta\beta\mathfrak{B}xxy + 12\beta(2\alpha\mathfrak{A} - \gamma\mathfrak{B})xyy\\ &+ 2\beta\beta\mathfrak{C}xx + 8\beta(3\alpha\mathfrak{B} - \gamma\mathfrak{C})xy + 2(6\alpha\alpha\mathfrak{A} - 6\alpha\gamma\mathfrak{B} + \gamma\gamma\mathfrak{C})yy\\ &+ 4\beta(\alpha\mathfrak{C} - 3\gamma\mathfrak{D})x + 4(3\alpha\alpha\mathfrak{B} - 2\alpha\gamma\mathfrak{C} + 3\gamma\gamma\mathfrak{D})y + 2(\alpha\alpha\mathfrak{C} - 6\alpha\gamma\mathfrak{D} + 6\gamma\gamma\mathfrak{E}).\end{aligned}$$

18. Coaequentur nunc inter se termini homologi utriusque formae et sequentibus aequationibus erit satisfaciendum

$$\begin{array}{c|l} xxyy & \gamma\gamma A = \beta\beta\mathfrak{A},\\ xxy & 2\alpha\gamma A - \beta\gamma B = \beta\beta\mathfrak{B},\\ xyy & \gamma\gamma B = 2\alpha\beta\mathfrak{A} - \beta\gamma\mathfrak{B},\\ xx & 6\alpha\alpha A - 6\alpha\beta B + \beta\beta C = \beta\beta\mathfrak{C},\\ yy & \gamma\gamma C = 6\alpha\alpha\mathfrak{A} - 6\alpha\gamma\mathfrak{B} + \gamma\gamma\mathfrak{C},\\ xy & 3\alpha\gamma B - \beta\gamma C = 3\alpha\beta\mathfrak{B} - \beta\gamma\mathfrak{C},\\ x & 3\alpha\alpha B - 2\alpha\beta C + 3\beta\beta D = \alpha\beta\mathfrak{C} - 3\beta\gamma\mathfrak{D},\\ y & \alpha\gamma C - 3\beta\gamma D = 3\alpha\alpha\mathfrak{B} - 2\alpha\gamma\mathfrak{C} + 3\gamma\gamma\mathfrak{D},\\ 1 & \alpha\alpha C - 6\alpha\beta D + 6\beta\beta E = \alpha\alpha\mathfrak{C} - 6\alpha\gamma\mathfrak{D} + 6\gamma\gamma\mathfrak{E}.\end{array}$$

Tres autem primae aequationes tantum duas dant determinationes

$$\beta = \frac{2\alpha A\sqrt{\mathfrak{A}}}{B\sqrt{\mathfrak{A}} + \mathfrak{B}\sqrt{A}} \quad \text{et} \quad \gamma = \frac{2\alpha\mathfrak{A}\sqrt{A}}{B\sqrt{\mathfrak{A}} + \mathfrak{B}\sqrt{A}},$$

quarta et quinta itidem unicam determinationem suppeditant

$$C - \mathfrak{C} = \frac{3(\mathfrak{A}BB - A\mathfrak{B}\mathfrak{B})}{2A\mathfrak{A}} = \frac{3}{2}\left(\frac{BB}{A} - \frac{\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}\right),$$

quae eadem quoque ex sexta sequitur; statuatur ergo

$$C = \frac{3BB}{2A} + n \quad \text{et} \quad \mathfrak{C} = \frac{3\mathfrak{B}\mathfrak{B}}{2\mathfrak{A}} + n.$$

Septima et octava etiam unicam determinationem involvunt

$$\frac{D\sqrt{A}+\mathfrak{D}\sqrt{\mathfrak{A}}}{B\sqrt{\mathfrak{A}}+\mathfrak{B}\sqrt{A}}=\frac{A\mathfrak{B}\mathfrak{B}+\mathfrak{A}BB-B\mathfrak{B}\sqrt{A\mathfrak{A}}+2nA\mathfrak{A}}{4A\mathfrak{A}\sqrt{A\mathfrak{A}}}$$

vel

$$D\sqrt{A}+\mathfrak{D}\sqrt{\mathfrak{A}}=\frac{B^3}{4A\sqrt{A}}+\frac{\mathfrak{B}^3}{4\mathfrak{A}\sqrt{\mathfrak{A}}}+\frac{nB}{2\sqrt{A}}+\frac{n\mathfrak{B}}{2\sqrt{\mathfrak{A}}};$$

statuatur ergo

$$D=\frac{B^3}{4AA}+\frac{nB}{2A}+\frac{m}{2\sqrt{A}} \quad \text{et} \quad \mathfrak{D}=\frac{\mathfrak{B}^3}{4\mathfrak{A}\mathfrak{A}}+\frac{n\mathfrak{B}}{2\mathfrak{A}}-\frac{m}{2\sqrt{\mathfrak{A}}},$$

qui valores in ultima aequatione substituti praebent

$$24(AE-\mathfrak{A}\mathfrak{E})=\frac{3B^4}{2AA}+\frac{6nBB}{A}+\frac{12mB}{\sqrt{A}}-\frac{3\mathfrak{B}^4}{2\mathfrak{A}\mathfrak{A}}-\frac{6n\mathfrak{B}\mathfrak{B}}{\mathfrak{A}}+\frac{12m\mathfrak{B}}{\sqrt{\mathfrak{A}}},$$

quare commode statui licebit

$$E=\frac{B^4}{16A^3}+\frac{nBB}{4AA}+\frac{mB}{2A\sqrt{A}}+\frac{l}{A},$$

$$\mathfrak{E}=\frac{\mathfrak{B}^4}{16\mathfrak{A}^3}+\frac{n\mathfrak{B}\mathfrak{B}}{4\mathfrak{A}\mathfrak{A}}-\frac{m\mathfrak{B}}{2\mathfrak{A}\sqrt{\mathfrak{A}}}+\frac{l}{\mathfrak{A}}.$$

19. Cum autem sumserimus $V=\frac{1}{(\alpha+\beta x+\gamma y)^2}$, erit

$$Q=\frac{-4\beta(Ax^4+2Bx^3+Cxx+2Dx+E)}{(\alpha+\beta x+\gamma y)^3}+\frac{2(2Ax^3+3Bxx+Cx+D)}{(\alpha+\beta x+\gamma y)^2},$$

$$P=\frac{-4\gamma(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}{(\alpha+\beta x+\gamma y)^3}+\frac{2(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})}{(\alpha+\beta x+\gamma y)^2}$$

sive

$$Q=\frac{2\gamma y(2Ax^3+3Bxx+Cx+D)+2(2\alpha A-\beta B)x^3+2(3\alpha B-\beta C)xx+2(\alpha C-3\beta D)x+2(\alpha D-2\beta E)}{(\alpha+\beta x+\gamma y)^3},$$

$$P=\frac{2\beta x(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})+2(2\alpha\mathfrak{A}-\gamma\mathfrak{B})y^3+2(3\alpha\mathfrak{B}-\gamma\mathfrak{C})yy+2(\alpha\mathfrak{C}-3\gamma\mathfrak{D})y+2(\alpha\mathfrak{D}-2\gamma\mathfrak{E})}{(\alpha+\beta x+\gamma y)^3},$$

unde investigari oportet integrale formulae $Pdx+Qdy$; ad quod si deinceps addatur $\frac{2\sqrt{XY}}{(\alpha+\beta x+\gamma y)^2}$, aggregatum quantitati constanti aequatum exhibebit

integrale completum aequationis

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0.$$

Pro illo autem integrali inveniendo ex prioribus valoribus pro P et Q exhibitis notetur fore separatim

$$\int Q\,dy=\frac{2\beta(Ax^4+2Bx^3+Cxx+2Dx+E)}{\gamma(\alpha+\beta x+\gamma y)^2}-\frac{2(2Ax^3+3Bxx+Cx+D)}{\gamma(\alpha+\beta x+\gamma y)}+\Gamma:x,$$

$$\int P\,dx=\frac{2\gamma(\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E})}{\beta(\alpha+\beta x+\gamma y)^2}-\frac{2(2\mathfrak{A}y^3+3\mathfrak{B}yy+\mathfrak{C}y+\mathfrak{D})}{\beta(\alpha+\beta x+\gamma y)}+\Delta:y,$$

quae duae expressiones aequales esse debent; quem in finem ponatur

$$\Gamma:x=\frac{2(Axx+Bx+N)}{\beta\gamma}\quad\text{et}\quad\Delta:y=\frac{2(\mathfrak{A}yy+\mathfrak{B}y+\mathfrak{N})}{\beta\gamma}$$

fietque

$\frac{1}{2}\beta\gamma(\alpha+\beta x+\gamma y)^2\int Q\,dy$	$\frac{1}{2}\beta\gamma(\alpha+\beta x+\gamma y)^2\int P\,dx$
$+A\gamma\gamma xxyy$	$+\mathfrak{A}\beta\beta xxyy$
$+B\gamma\gamma xyy$	$+\beta(2\mathfrak{A}\alpha-\mathfrak{B}\gamma)xyy$
$+\gamma(2A\alpha-B\beta)xxy$	$+\mathfrak{B}\beta\beta xxy$
$+N\gamma\gamma yy$	$+(\mathfrak{A}\alpha\alpha-\mathfrak{B}\alpha\gamma+\mathfrak{N}\gamma\gamma)yy$
$+(A\alpha\alpha-B\alpha\beta+N\beta\beta)xx$	$+\mathfrak{N}\beta\beta xx$
$+\gamma(2B\alpha-C\beta+2N\beta)xy$	$+\beta(2\mathfrak{B}\alpha-\mathfrak{C}\gamma+2\mathfrak{N}\gamma)xy$
$+\gamma(2N\alpha-D\beta)y$	$+(\mathfrak{B}\alpha\alpha-\mathfrak{C}\alpha\gamma+\mathfrak{D}\gamma\gamma+2\mathfrak{N}\alpha\gamma)y$
$+(B\alpha\alpha-C\alpha\beta+D\beta\beta+2N\alpha\beta)x$	$+\beta(2\mathfrak{N}\alpha-\mathfrak{D}\gamma)x$
$+E\beta\beta-D\alpha\beta+N\alpha\alpha$	$+\mathfrak{E}\gamma\gamma-\mathfrak{D}\alpha\gamma+\mathfrak{N}\alpha\alpha.$

20. Hae conditiones cum praecedentibus (§ 18) perfecte conveniunt, si modo sumatur

$$N=\tfrac{1}{6}C\quad\text{et}\quad\mathfrak{N}=\tfrac{1}{6}\mathfrak{C}.$$

Dividamus singulos terminos per $\beta\gamma$, ut prodeat valor formulae

$$\tfrac{1}{2}(\alpha+\beta x+\gamma y)^2\int Q\,dy,$$

qui substitutis valoribus ante inventis reperietur

$$xxyy\sqrt{A\mathfrak{A}} + Bxyy\sqrt{\frac{\mathfrak{A}}{A}} + \mathfrak{B}xxy\sqrt{\frac{A}{\mathfrak{A}}}$$
$$+\frac{1}{6}Cyy\sqrt{\frac{\mathfrak{A}}{A}} + \frac{1}{6}\mathfrak{C}xx\sqrt{\frac{A}{\mathfrak{A}}} + \left(\frac{B\mathfrak{B}}{\sqrt{A\mathfrak{A}}} - \frac{2}{3}n\right)xy$$
$$+\left(\frac{BB\mathfrak{B}}{4A\sqrt{A\mathfrak{A}}} - \frac{nB}{3A} + \frac{n\mathfrak{B}}{6\sqrt{A\mathfrak{A}}} - \frac{m}{2\sqrt{A}}\right)y + \left(\frac{B\mathfrak{B}\mathfrak{B}}{4\mathfrak{A}\sqrt{A\mathfrak{A}}} - \frac{n\mathfrak{B}}{3\mathfrak{A}} + \frac{nB}{6\sqrt{A\mathfrak{A}}} + \frac{m}{2\sqrt{\mathfrak{A}}}\right)x$$
$$+\frac{BB\mathfrak{B}\mathfrak{B}}{16A\mathfrak{A}\sqrt{A\mathfrak{A}}} + \frac{n(B\sqrt{\mathfrak{A}} + \mathfrak{B}\sqrt{A})^2}{24A\mathfrak{A}\sqrt{A\mathfrak{A}}} - \frac{nB\mathfrak{B}}{4A\mathfrak{A}} + \frac{m(B\sqrt{\mathfrak{A}} - \mathfrak{B}\sqrt{A})}{4A\mathfrak{A}} + \frac{l}{\sqrt{A\mathfrak{A}}}.$$

Sit haec forma brevitatis gratia $= S$ eritque integrale completum

$$\frac{S+\sqrt{XY}}{(\alpha+\beta x+\gamma y)^2} = \text{Const.}$$

seu

$$S+\sqrt{XY} = \text{Const.}(B\sqrt{\mathfrak{A}} + \mathfrak{B}\sqrt{A} + 2Ax\sqrt{\mathfrak{A}} + 2\mathfrak{A}y\sqrt{A})^2,$$

quod etiam hac forma concinniori exhiberi potest

$$S+\sqrt{XY} = \text{Const.}\left(\frac{B}{\sqrt{A}} + \frac{\mathfrak{B}}{\sqrt{\mathfrak{A}}} + 2x\sqrt{A} + 2y\sqrt{\mathfrak{A}}\right)^2.$$

Quare, dum functiones X et Y conditionibus ante definitis sint praeditae, hoc modo habebitur integrale completum aequationis differentialis

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0.$$

21. Haec investigatio aliquanto generalius institui potest tribuendo ipsi V talem valorem

$$\frac{1}{(\alpha+\beta x+\gamma y+\delta xy)^2};$$

quo facilius autem calculi molestias superare queamus, observo, dummodo variabiles x et y quantitate constante augeantur vel minuantur, eum ad hanc formam $\frac{1}{(\alpha+xy)^2}$ reduci posse; expedito autem calculo restitutio facile instituetur.

Considerabo ergo hanc aequationis differentialis formam

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,$$

quam integrabilem reddi assumo ope multiplicatoris $P\sqrt{X}+Q\sqrt{Y}$, ut integrari debeat haec formula

$$Pdx+Qdy+\frac{Qdx\sqrt{Y}}{\sqrt{X}}+\frac{Pdy\sqrt{X}}{\sqrt{Y}}=0.$$

Statuatur partis posterioris integrale $=2V\sqrt{XY}$ fietque, ut vidimus,

$$Q=2X\left(\frac{dV}{dx}\right)+V\frac{dX}{dx} \quad \text{et} \quad P=2Y\left(\frac{dV}{dy}\right)+V\frac{dY}{dy}.$$

Sit igitur $V=\frac{1}{(a+xy)^2}$ ideoque

$$\left(\frac{dV}{dx}\right)=\frac{-2y}{(a+xy)^3} \quad \text{et} \quad \left(\frac{dV}{dy}\right)=\frac{-2x}{(a+xy)^3},$$

ita ut habeamus

$$Q=\frac{-4Xy}{(a+xy)^3}+\frac{dX}{dx}\frac{1}{(a+xy)^2} \quad \text{et} \quad P=\frac{-4Yx}{(a+xy)^3}+\frac{dY}{dy}\frac{1}{(a+xy)^2}.$$

Nunc autem effici debet, ut formula $Pdx+Qdy$ integrationem admittat; hunc in finem duplici modo eius integrale capiatur, dum vel y vel x constans accipitur, sicque obtinebimus

$$\int Pdx=\frac{4Y}{yy(a+xy)}-\frac{2aY}{yy(a+xy)^2}-\frac{dY}{ydy}\cdot\frac{1}{a+xy}+\frac{\Gamma:y}{yy},$$
$$\int Qdy=\frac{4X}{xx(a+xy)}-\frac{2aX}{xx(a+xy)^2}-\frac{dX}{xdx}\cdot\frac{1}{a+xy}+\frac{\Delta:x}{xx},$$

quas duas formas inter se aequales reddi oportet. Multiplicando ergo per $xxyy(a+xy)^2$ habebimus

$$4xxY(a+xy)-2axxY-\frac{xxydY}{dy}(a+xy)+xx\Gamma:y\cdot(a+xy)^2$$
$$=4yyX(a+xy)-2ayyX-\frac{xyydX}{dx}(a+xy)+yy\Delta:x\cdot(a+xy)^2,$$

unde fingamus

$$X=Ax^4+2Bx^3+Cxx+2Dx+E, \quad \Delta:x=Lxx+Mx+N,$$
$$Y=\mathfrak{A}y^4+2\mathfrak{B}y^3+\mathfrak{C}yy+2\mathfrak{D}y+\mathfrak{E}, \quad \Gamma:y=\mathfrak{L}yy+\mathfrak{M}y+\mathfrak{N},$$
$$\frac{dX}{dx}=4Ax^3+6Bxx+2Cx+2D \quad \text{et} \quad \frac{dY}{dy}=4\mathfrak{A}y^3+6\mathfrak{B}yy+2\mathfrak{C}y+2\mathfrak{D}.$$

Hinc nostrae expressiones induent has formas

$xxyy(a+xy)^2\int Qdy$	$xxyy(a+xy)^2\int Pdx$
$+Lx^4y^4$	$+\mathfrak{L}x^4y^4$
$+Mx^3y^4$	$+2\mathfrak{B}x^3y^4$
$+2Bx^4y^3$	$+\mathfrak{M}x^4y^3$
$+Nxxy^4$	$-2a\mathfrak{A}xxy^4$
$+2(C+aL)x^3y^3$	$+2(\mathfrak{C}+a\mathfrak{L})x^3y^3$
$-2aAx^4yy$	$+\mathfrak{N}x^4yy$
$+2(3D+aM)xxy^3$	$-2a\mathfrak{B}xxy^3$
$-2aBx^3yy$	$+2(3\mathfrak{D}+a\mathfrak{M})x^3yy$
$+aaLxxyy$	$+aa\mathfrak{L}xxyy$
$+2(2E+aN)xy^3$	$+0xy^3$
$+0x^3y$	$+2(2\mathfrak{E}+a\mathfrak{N})x^3y$
$+(2aD+aaM)xyy$	$+0xyy$
$+0xxy$	$+(2a\mathfrak{D}+aa\mathfrak{M})xxy$
$+(2aE+aaN)yy$	$+0yy$
$+0xx$	$+(2a\mathfrak{E}+aa\mathfrak{N})xx.$

22. Harum formarum coaequatio suppeditat sequentes determinationes

$$\mathfrak{L}=L,\quad M=2\mathfrak{B},\quad \mathfrak{M}=2B,\quad N=-2a\mathfrak{A},\quad \mathfrak{N}=-2aA,$$
$$\mathfrak{C}=C,\quad D=-a\mathfrak{B},\quad \mathfrak{D}=-aB,\quad E=aa\mathfrak{A},\quad \mathfrak{E}=aaA,$$

ita ut habeatur haec aequatio differentialis

$$\frac{dx}{\sqrt{(Ax^4+2Bx^3+Cxx+2Dx+E)}}+\frac{dy}{\sqrt{\left(\frac{E}{aa}y^4-\frac{2D}{a}y^3+Cyy-2aBy+aaA\right)}}=0,$$

cuius integrale completum est

$$\frac{2Bxxy-\frac{2D}{a}xyy-2aAxx-\frac{2E}{a}yy+2Cxy-2aBx+2Dy+2\sqrt{XY}}{(a+xy)^2}=\text{Const.}$$